ThinQ

think • inquire • question

CONSTRUCTING THEORIES

OF GEOMETRY WITH EXCURSIONS INTO BIOLOGY*

K P Mohanan and *Tara Mohanan*

25 February 2024

* This work is part of

THE FIG TREE PROJECT

funded by

the late Dr Sunita Anne Abraham

PhD, University of Birmingham, UK 1993

NOTE: If you find any errors, please alert us at:

tara.mohanan@gmail.com and mohanan.kp@gmail.com

ACKNOWLEDGEMENTS

This booklet owes its existence to the generous gift to ThinQ from our dear friend, Sunita Abraham. While Sunita is no longer with us, and we have no words to express our deep gratitude to her, she will live on in the project that was born out of her gift — the project of creating learning materials for children — the Fig Tree Project.

We are grateful to Malavika Mohanan, Sriram Naganathan, Aditi Ahuja, and Rashmi Jejurikar, for extensive feedback on an earlier draft of this booklet, with valuable suggestions for making it better.

TO THE READER

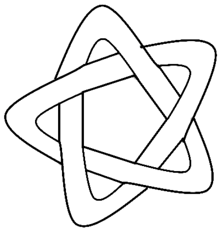
This book is an attempt to introduce you to the art and craft of constructing and evaluating theories. We use geometry as the playground for learning, with occasional excursions into biology. Geometry and biology are different in their bodies of knowledge, and yet, they have a set of shared methodological strategies and techniques for constructing theories. Our focus will be on those strategies and techniques, and higher-order cognitive capacities that are relevant across different domains. So this is not a textbook that aims to 'teach' the 'facts' of geometry or of biology.

We expect the book to be useful for high school students, as well as college students regardless of their chosen discipline. We have tried to write it in a way that 8^{th} grade students would find it somewhat accessible, and interesting. We hope you will enjoy the book, and also benefit from reading it.

And yes, to make the most of it, do try to work through all the exercises. ☺

CONSTRUCTING THEORIES OF GEOMETRY WITH EXCURSIONS INTO BIOLOGY

CONTENTS



CHAPTER 1:
A THEORY OF TRIANGLES

1.1 What is a Theory?

The goal of this book is to help learners develop the capacity to construct theories in any domain of academic knowledge — whether it is mathematical theories, scientific theories, ethical theories, or some other type of theories. To really do justice to that grand goal, we would need at least a thousand pages, and several courses for students to learn from. So we will begin with something small — constructing a theory of triangles, and within that, constructing a theory of right-angled triangles, and see where that leads us.

But to do that, we need to say something about what we mean by the word *theory*. Now, if we simply tell you what a theory is, you would hear or see the words and sentences we utter or write, and perhaps form a vague image in your mind. But to get a sense of what a theory is, you need to go through the experience of constructing theories a few times.

We will still say something about theories anyway, with the hope that half way through this book, you will return to what is said here, and get a clearer understanding, and that at the end of the book, you will return to these sentences again, and say, "Aha! Now I understand what a theory is."

So, what is a theory? Here is the first approximation:

> *A theory is a set of statements from which*
> *we can arrive at a set of conclusions.*

We can hear you say, "An example please!" Okay, here is an example. Consider these two statements (S):

- S1: Animate bodies have the capacity to move on their own.
- S2: Inanimate bodies do not have the capacity to move on their own.

Given S2, together with our knowledge that dead mice are inanimate (i.e., not living), it is legitimate to conclude that dead mice do not move on their own. And given S1, along with our knowledge that living mice are animate, it is legitimate to conclude that living mice can move on their own.

What does 'move on their own" mean? It means that if we place a living mouse on a table, we may not find it in the same place after a while. But if we place a dead mouse on the table, it would be where we left it earlier, unless a live cat runs away with it, of course. (Hmm! What do you think would happen if you leave a dead mouse near a live cat?)

The conclusions from S1 and S2 are not restricted to just dead and living mice and cats. They also apply to other inanimate bodies like stones, rocks, cups, pens, and chairs, and animate bodies like cows, crows, cobras, flies, and fish. They form a tentative rudimentary theory of the motion — change of location — of animate and inanimate things in the world.

Does this theory work? To answer that question, we need to look at other examples. Suppose we dig out a small plant from the garden, and leave it on the table. If we were to go back to the table an hour later, do you think the plant would still be where we left it? Or would it have moved to some other location, without some other organism moving it? If your answer is it would not have moved on its own, the next question is: Are plants animate or inanimate? If we treat them as animate, we have a problem. According to our theory, if plants are animate, they would move on their own, Our experience tells us that they cannot. We therefore need to modify our theory of motion such that its conclusions are not in conflict with our experience.

That should be sufficient for now for an approximate understanding of the concept of theories. So let us move on to practice the art and craft of theory construction with triangles in the terrain of geometry.

[Note: Some of you might find the new vocabulary here daunting or scary. We urge you to try to understand the essence of it, and read on without being dissuaded by words you don't understand yet. ☺]

1.2 Describing Right-Angled Triangles

Let us begin our journey by looking at some triangles.

You may have come across the term 'Right-Angled Triangles' before. We will call them RATs. Here are a few examples of RATs:

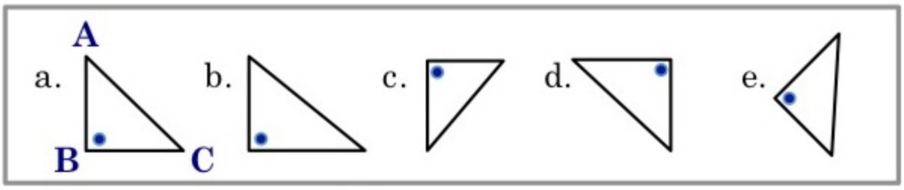


Figure 1-1

What are the properties of RATs? Let's make a list:

1) a) A RAT is a closed figure.

b) A RAT has three angles.

c) A RAT has three straight lines.

d) One of the angles of a RAT is a right angle (blue dot in *Fig. 1-1*).

e) The length of the square of the side opposite the right angle is equal to the sum of the squares of the lengths of the other two sides.
For example, in *Fig. 1-1a*, the three sides are AB, BC, and CA.
AC^2 is equal to the sum of AB^2 and BC^2. That is: $AC^2 = AB^2 + BC^2$

f) The area of a RAT is equal to the length of a side adjacent to the right angle, multiplied by the length of the other adjacent side, divided by 2.

For example, the area of the RAT in *Fig. 1-1a* is: $\frac{1}{2} AB \times BC$

1.3 Equilateral Triangles

Let us move on to a different geometric figure. Here are a few examples of Equilateral Triangles (ETs):

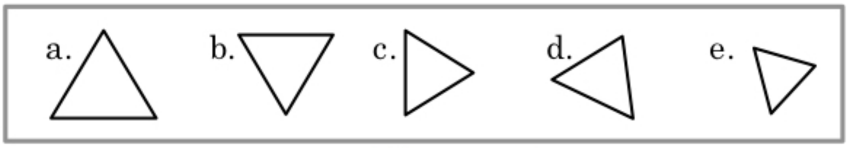


Figure 1-2

Notice that the figures in *Fig. 1-1* are not ETs, and that those in *Fig. 1-2* are not RATs.

If we look at the properties of ETs, we find:

2) a) An ET is a closed figure.
b) An ET has three angles.
c) An ET has three straight lines.
d) In an ET, all the angles are of the same size.
e) In an ET, all the sides are of the same size.
f) The area of an ET is equal to the length of one of its sides multiplied by the length of the perpendicular from the angle opposite that side, divided by 2.

For example, in this figure, take side *BC*. The angle opposite *BC* is A. The perpendicular from A to *BC* is *AD*. So,

the area of the ET is: $\frac{BC \times AD}{2}$

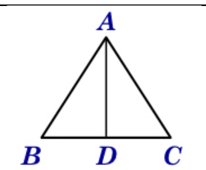


Figure 1-2 f

Looking at 1a-f and 2a-f, do you find we're being repetitive?

When we look at *Fig. 1-1a* and *Fig.1-2f*, the two look very different. In RATs, we multiply two sides for calculating their area; while in ETs, we multiply a side and the perpendicular. And yet, we use the same formula for both. When we look closer, we see why: AB in *Fig. 1-1a* and AD in *Fig. 1-2f* are both perpendicular to BC, the base.

1.4 Triangles

One way to deal with the unnecessary repetition above is to note that RATs and ETs are both subcategories of the category TRIANGLE. Examples of triangles include not only those in Figs. 1-1 and 1-2, but also those in *Fig. 1-3*:

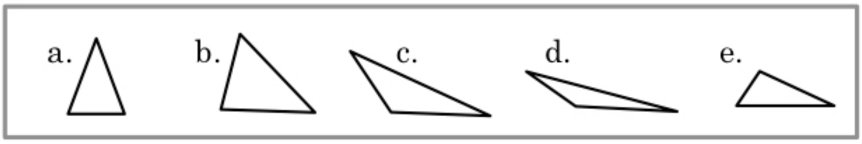


Figure 1-3

What about the properties of Triangles? Here they are:

3) a) A Triangle is a closed figure.

b) A Triangle has three angles.

c) A Triangle has three straight lines.

d) In a Triangle, the sum of the angles is equal to two right angles (= 180º).

e) The area of a Triangle is equal to the length of one of its sides multiplied by the length of the perpendicular from the angle opposite that side, divided by 2.

Given (3a-c), you would agree that repeating the statements as (1a-c) and (2a-c) is unnecessary, or ***redundant***. This is because since RATs and ETs are triangles, it follows that whatever is true of triangles would be true of RATs and ETs as well. Let's take a closer look at what is behind that statement.

1.5 Derivation

How do we know that the statements in (1a-b) and (2a-b) *follow logically from* (3a-b)? By showing that (1a-b) and (2a-b) can be *derived from* (3a-b).

This requires a general principle about the subcategories of a category:

4) General Principle of Logical Inheritance

All the properties of a category are inherited by its subcategories.

At the beginning of section 1.4, we noted that:

5) The categories 'RAT' and 'ET' are subcategories of the category 'Triangle'.

Given (5), the *general principle of logical inheritance* in (4) allows us to *derive* (1a-c) and (2a-c) from (3a-c). The derivation given below illustrates the application of this general principle. [Note that in the derivation, C1 and C2 stand for Conclusion-1 and Conclusion-2 respectively.

6) Derivation

A Triangle has three angles. (3b)

The categories 'RAT' and 'ET' are subcategories of the category 'TRIANGLE '. (5)

The properties of a category are inherited by its subcategories. (4)

RATs and ETs inherit all the properties of the category 'TRIANGLE'. (C1)

Hence, a RAT and an ET have three angles (C2): (1a), (2a)

We conclude from this derivation (from 3b and C1) that RATs and ETs have three angles. In arriving at this conclusion, we made an *inference* based on the statements we had accepted.

The process by which we arrive at such inferences is called ***reasoning***.

Exercise 1

Derive (1c) and (2c) from (3c). Write down ALL the steps of reasoning as we have done in the derivation in (6) above.

1.6 The Structure of Theories

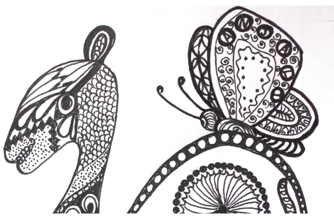
In sections 1.2 and 1.3, we began with descriptions of RATs and ETs. In the next two sections, we were developing a rudimentary theory. The difference between a theory of X and a description of X is this:

> While a *description* of X is a collection of statements about X,
> a *theory* of X provides a logical structure to our knowledge,
> using derivations of the kind illustrated in (6) above.

1.6.1 Reasoning and Derivations

A derivation is a sequence of steps of reasoning that allows us to infer something based on something else that we take to be true. For example, suppose we take the following statements to be true.

1. Athena is taller than Apollo.
2. Socrates is taller than Aristotle.
3. Apollo is taller than Zeno.
4. Plato is taller than Socrates.
5. Zeno is taller than Plato.



Now we are asked: Who is taller, Athena or Aristotle?

With some effort, we should be able to infer the answer:

Athena is taller than Aristotle.

Let us work through the steps that we intuitively followed to arrive at this answer. In the following derivation, P stands for 'premise', C stands for 'conclusion', and the underlined parts in blue — "from x to y" — explicitly indicate the steps of the derivation:

P1. Athena is taller than Apollo.

P2. Apollo is taller than Zeno.
Therefore, Athena is taller than Zeno — C1: **from 1 and 3**

P3. Zeno is taller than Plato.
Therefore, Athena is taller than Plato. — C2: **from 5 and C1**

P4. Plato is taller than Socrates.
Therefore, Athena is taller than Socrates. — C3: **from 4 and C2**

P5. Socrates is taller than Aristotle.
Therefore, Athena is taller than Aristotle. — C4: **from 2 and C3**

This is an example of what an explicit process of reasoning looks like. When we explore reasoning, as in this example, we use the following terms:

Premise (P): the statements that we take to be true;

Conclusion (C): the statement that we infer from the premises; and

Derivation (D): the steps that connect the premises and the final conclusion.

The statements in P1-P5 are the premises, C1–C4 are the conclusions, and of these, C4 is the final conclusion. The structure of these concepts, which we will refer to as the PDC structure, can be pictured as in *Fig. 1-4*:

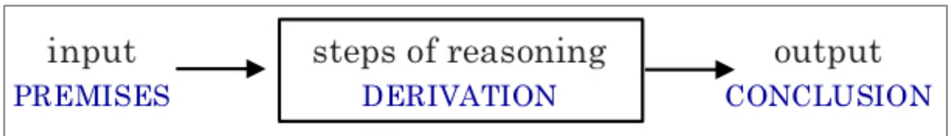


Figure 1-4

We will come back to these ideas in greater detail in Chapter 3.

1.6.2 Definitions, Postulates, and Common Notions

Geometry as a branch of mathematics was founded by ***Euclid*** (325 – 265 BCE), a Greek mathematician, known as the 'father of geometry'. Euclid distinguishes between three kinds of premises: definitions, postulates, and common notions.

The ***definition*** (DEF) of a concept tells us what comes under that concept and what does not. For example, a definition of *even* numbers tells us which numbers are even and which ones are not. A definition of *mammals* tells us which animals are mammals and which ones are not. A definition of *democratic systems* tells us which systems are democratic and which ones are not.

Here are some examples of definitions in geometry:

Right Angle (DEF): An angle that is formed by two lines that are perpendicular to each other.

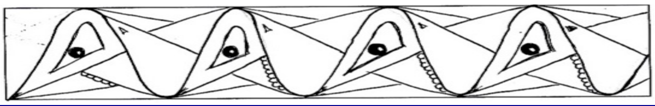
Obtuse Angle (DEF): An angle whose size is greater than that of a Right Angle.

Acute Angle (DEF): An angle whose size is less than that of a Right Angle.

Now, what is the definition of 'perpendicular', a concept that appears in the definition of a Right Angle?

Perpendicular (DEF): Given two straight lines AB and CD, with C on AB, CD is perpendicular to AB if angles ACD and BCD are congruent.

Congruent (DEF): If two geometric objects are the same shape and size, and when we place one on top of the other, they match exactly, we say that they are congruent.



DEFINITIONS: THEIR FUNCTION

The definition of a concept or a category allows us to determine whether or not a given candidate comes under that concept/category.

For instance, we treat the numbers 2, 4, 6, 8… as *even integers*, and 3, 5, 7… as *not even integers*. Consider this definition of *even integer*:

> ***Even integer*** (DEF): An even integer is one that can be divided by two, yielding an integer without any remainder.

Given any integer, this definition allows us to determine whether or not it is even. To take another example, consider this definition of *birds*:

> ***Bird*** (DEF): An animal that has a beak.

Given this definition, it follows that parrots and crows are birds because they have beaks, while cats and squirrels are not birds, because they don't have beaks.

It might appear that there is a countereexample to this definition in the duck-billed platypus (https://en.wikipedia.org/wiki/Platypus). Now, if a beak is an organ made out of a hard shell, then a bill is not a beak, it is a snout. Therefore, a platypus is not a counterexample.

1.6.3 Axioms and Axiomatic Systems

As mentioned earlier, the study of geometry as a branch of mathematics was initiated by Euclid. In his book called *Elements* (https://en.wikipedia.org/wiki/%27s_Elements), he used the term ***postulates*** to refer to premises that are specific to geometry. Here are some examples of postulates:

Postulate 1: Given any two distinct points, one and only one straight line exists between them.

Postulate 2: Any finite straight line can be extended indefinitely as a straight line.

The term ***common notion*** in Euclid's *Elements* denotes premises that apply to mathematics in general, not just to geometry. Here are some examples of common notions:

Common Notion 1: Things that are equal to the same thing are also equal to one another.

Common Notion 2: If equal things are added to equal things, then the results are equal.

Let us accept the essence of the Euclidean idea of postulates and common notions. We can call them ***axioms***, postulates being geometry-specific axioms, and common notions being mathematics-specific axioms. We can generalise further, making a distinction between *discipline-specific axioms* and *general axioms* as follows:

Discipline-specific Axioms: axioms specific to a particular discipline.

General Axioms: axioms that are relevant in all disciplines and discipline groups: mathematics, the sciences, history, psychology, sociology, anthropology, linguistics, economics, philosophy, and so on.

> Suppose we are inquiring into ethical issues, and we accept the following statement as a starting point, which we take as an axiom, as we cannot justify (give explicit reasons for accepting) it:
>
> **Axiom**: Causing harm to others is unethical.
>
> Now suppose Zeno tortured a cat.
>
> Given that torturing causes harm, we can take the act of torturing as a subcategory of the category of acts of causing harm. And it follows from the axiom that what Zeno did is unethical.
>
> Now, while the axiom about causing harm to others is specific to ethical inquiry, the Principle of Logical Inheritance in subcategorisation [Section 1.5] is an example of a General Axiom. It applies in all domains of inquiry.

Euclid saw axioms ('postulates' and 'common notions' for him) as 'self-evident truths'. However, subsequent developments in mathematics show that all axioms are ***assumptions***. This means that, even though some of them may appear to be self-evident (obvious) to people, they are not self-evident truths. They are premises that function as starting points for particular theories.

A system of knowledge that exhibits the property of deriving conclusions from axioms and definitions is called an ***axiomatic system***. Any fully fleshed-out theory, whether in mathematics, the physical-biological-human sciences, or the humanities includes an axiomatic system.

Exercise 2

a. Group the statements in (1), (2), and (3) (on pages 8-10) into premises and conclusions.

b. Come up with a derivation for each conclusion you identified in task (a).

   As you worked through the derivations, did you make use for any other statements not listed here? Were these statements definitions or axioms?

c. Separate ALL the premises into definitions and axioms.

d. Separate the axioms into General Axioms and Discipline-Specific Axioms.

Note: The term 'discipline-specific' in (d) is a bit vague. We do not have a clear idea of what counts as a *discipline*, a *field*, or a *discipline group,* and what does not! For example, is Zoology a *discipline* or a *field* within the discipline of biology? Is molecular biology a discipline, or a field within a discipline called life sciences? Does the study of society (covering humans and say, bees) form a single discipline? If they are two separate disciplines, do they form part of a single discipline group?

Despite this vagueness, it is still useful to invest some time on the task (d).

1.7 Theories vs. Descriptions

As mentioned earlier, the statements about triangles in (1), (2), and (3) (Sections 1.1–1.4, on pages 8-10) are descriptions, not theories. A description is a collection of statements about a given entity. The statements in (7) below form the description of a human being called Zeno:

7)
a. Zeno is a human being.
b. Zeno is an adult.
c. Zeno is a living organism.
d. Zeno has eukaryotic cells.
e. Zeno has two eyes.
f. Zeno has a mouth.
g. Zeno has a heart.
h. Zeno has vertebrae.
j. Zeno has bones.
k. Zeno is male.
l. Zeno is 180 cm tall.
m. Zeno weighs 82 kilograms.
n. Zeno is diabetic.
o. Zeno is prone to depression.
p. Zeno works as a banker.
q. Zeno has two siblings.
r. Zeno is married to Athena.
s. Zeno has three children.

And the statements in (8) form part of a description of human beings:

8)
a. Humans are living organisms.
b. Humans have eukaryotic cells.
c. Humans have two eyes.
d. Humans have a mouth.
e. Humans have a heart.
f. Humans have vertebrae.
g. Humans have bones.

An important feature of a theory is that it is built on what is regular, and therefore predictable; it sets aside what is random, and therefore not predictable. By 'predictable', we mean something that can be inferred from some other information. For example, how do we infer that Zeno has eukaryotic cells (7d)? From the statements: Zeno is a human being (7a); and Humans have eukaryotic cells (8b).

Thus, a theory clearly outlines (a) the premises (axioms and definitions) of the theory, (b) the predictable conclusions that can be inferred or arrived at from these premises, and (c) the derivations from (a) to (b).

Exercise 3

a. Do a Google search for butterflies, ants, and insects. For each category, write down a set of sentences that describe their anatomy, along the lines illustrated in (1), (2), and (3) on Pg. 8-10. [To save time, do not go beyond what you judge to be the 10 most important points.]

b. Try to group the statements for each category into premises and conclusions in such a way that the number of premises is minimised.

c. Make sure that you have a derivation for each conclusion, clearly stating the steps of reasoning from the premises to the conclusion.

d. Separate the premises into definitions and axioms.

e. Separate the axioms into discipline-specific and general axioms.

In the Chapters to come, we will learn how to construct and evaluate theories by separating *what is not predictable* (premises) from *what is predictable* (conclusions), and deducing the conclusions from the premises.

While learning the basics of this strategy, we will also learn a number of other strategies, techniques, and norms of theory construction, and of knowledge construction in general.

*A Note on **Euclid***

As far as we know, Euclid in Ancient Greece and Panini in Ancient India originally developed theories in this sense, in mathematical and scientific inquiry. Panini's work, *Ashtadhyayi* (pronounced /aSTaadhyaayii/) in Sanskrit is not easy to read. But the English translation of Euclid's *Elements* is not too difficult to follow, if you are willing to invest time. It is downloadable at:

http://farside.ph.utexas.edu/Books/Euclid/Elements.pdf

[If you are interested in reading Euclid's *Elements* to understand reasoning in mathematics, it may be a good idea to read this book first.]

CHAPTER 2:
A THEORY OF QUADRILATERALS

2.1 Squares

Here are a few examples of Squares:

Figure 2-1

What are the properties of Squares? Let us make a list:

1) a) A Square is a closed figure.
b) A Square has four angles.
c) The four angles of a Square are equal in size.
d) A Square has four straight lines.
e) The four lines of a Square are equal in size.
f) The sum of angles in a Square is equal to four right angles.
g) The area of a Square is equal to the length of one of the sides multiplied by the length of another side (= the square of one of the sides).
h) If AC is the diagonal of a Square ABCD, then the square of AC is equal to twice the square of AB (that is, $AC^2 = AB^2 \times 2$).

Exercise 1

a) Based on your current knowledge of geometry, what more can you say about a Square? Add to the list in (1).

b) Expand your list even further. For example, take a look at these URLs:

"Properties of a rhombus, a rectangle, and a square"
(https://www.youtube.com/watch?v=3i2yp-lI_V4)

"Theorems dealing with rectangles, rhombuses, and squares"
(https://mathbitsnotebook.com/Geometry/Quadrilaterals/QDRectangle.html)

c) Based on your knowledge of geometry, make a list of the properties of Rectangles, Rhombuses, and Parallelograms. Based on the way we put together the properties of RATs, ETs, and Triangles in Chapter 1, try to put together the properties of Squares, Rectangles, Rhombuses, Parallelograms, and Quadrilaterals.

d) Set up category-subcategory relations among them in such a way that using the Principle of Logical Inheritance in Chapter 1, you can deduce as many properties of these geometrical objects as possible from a small number of premises.

See Section 2.4 for a worked out example of a derivation using the Principle of Logical Inheritance.

2.2 Triangles and Quadrilaterals

Chances are that the exercises in Section 2.1 led you to propose that Squares, Rectangles, Rhombuses, and Parallelograms are subcategories of Quadrilaterals.

Did you, by any chance, also consider that:

Rectangles are a subcategory of Parallelograms;
Squares are a subcategory of Rectangles; and
Squares are also a subcategory of Rhombuses?

If you hadn't thought of these possibilities, reflect on them now, and write down your subcategorisation statements.

We have explored the ***properties*** of both Triangles and Quadrilaterals. So now, we can ask questions about the ***relation*** between them. For example:

Is it possible to divide a Square into two Triangles?
If it is, what kind of Triangles do you get? RATs? ETs?
How about Isosceles Triangles (ITs)?
Is it possible to fill a square with RATs, without any gaps?
How about ETs?
How about ITs that are not equilateral?

Explore these relations, and try to make a list of statements that describe the patterns you discover. Then try to derive these relations from the premises you already have. This can be challenging, but still worth a try, without getting frustrated if you can't see any patterns.

Now consider Pentagons. Do you see a set of shared properties across Triangles, Squares, and Pentagons? Describe what these geometric objects have in common.

Exercise 2

In (1), we stated a number of properties of Squares. Suppose we remove one or more of the properties in (a)-(e). Would you still get the 'definition' of a Square? Or would you get geometric objects which are not Squares? Which category would these objects belong to?

Exercise 3

Just as we listed the properties of Squares, make a list of the properties of Quadrilaterals, Parallelograms, and Rectangles.

2.3 Logical Inheritance in Geometry and Biology

In Chapter 1, we saw a number of repetitions in the descriptions of RATs, ETs, and Triangles. To eliminate these repetitions, we did the following:

~ treated RATs and ETs as subcategories of the category 'TRIANGLE';

~ set up the Principle of Logical Inheritance in subcategorisation;

~ used this principle to avoid the duplication of specifications in the descriptions of RATs and ETs; and

~ derived the predictable statements (conclusions) from the premises.

Exercise 4

Use the Principle of Logical Inheritance to eliminate the redundant statements in:

a. the description of Parallelograms by deriving them from the properties of Quadrilaterals;

b. the description of Rectangles by deriving them from the properties of Parallelograms; and

c. the description of Squares by deriving them from the properties of Rectangles.

MINIMISING POSTULATES AND MAXIMISING THE RANGE OF CONCLUSIONS

Suppose we state all the properties of squares as premises. They would include:

A square has four angles.
A square has four straight lines. They are all equal.
A square has two diagonals. They are equal.
The area of a square is the product of the lengths of any of its two sides.

Now, these properties can be predicted if we make a simple assumption, that:

A square is a kind of rectangle.

If we state this assumption as a premise in our theory, it allows us to derive most properties of squares from the properties of rectangles. Many of the properties of rectangles can be derived in the same way, by assuming that a rectangle is a parallelogram, a parallelogram is a quadrilateral, and a quadrilateral is a polygon. This allows us to construct a theory of geometry that uses the *smallest number of postulates* to deduce *the largest range of conclusions*, which are the *logical consequences of those postulates.* (Such logical consequences are called *predictions* in science, and *theorems* in mathematics.)

ADDITIONAL EXERCISES

Exercise 5

a) Do an Internet search for ‘crow’, ‘vulture’, and ‘parrot’. Do the same for ‘bird’. For each category, write down a set of sentences that describe their anatomy, along the lines illustrated in Chapter 1, (1), (2), and (3) on Pg. 7-8. [To save time, do not go beyond what you judge to be the 10 most important points.]

b) Postulate category–subcategory relations among them.

c) Try to group the statements for each category into premises and conclusions in such a way that the number of premises is minimised.

c) Make sure that you have a derivation for each conclusion, clearly stating the steps of reasoning from the premises to the conclusion.

d) Try to separate the premises into definitions and axioms.

e) Try to separate the axioms into General Axioms and Discipline-Specific Axioms.

Exercise 6

a) Do an Internet search for the vertebrae of snakes, fish, and birds. For each category of vertebrae, write down a set of sentences that describe their structure. In the interest of time, do not go beyond what you judge to be the FIVE most important points.

b) Try to group the statements for each category of vertebrae into premises and conclusions in such a way that the number of premises is minimised.

c) Make sure that you have a derivation for each conclusion.

d) Try to separate the premises into definitions and axioms.

e) Try to separate the axioms into General Axioms and Discipline-Specific Axioms.

Exercise 7

In Aristotle’s categorisation of animate entities, the first division was human vs. non-human. The category ‘non-human’ was further divided into ‘animal’ and ‘plant’.

The classification proposed by Carl Linnaeus was a bit different. In it, the first division was animal vs. plant. Humans came under animals.

A third possible categorisation would be human, animal, and plant. The three possibilities are given in *Fig. 2-2* below:

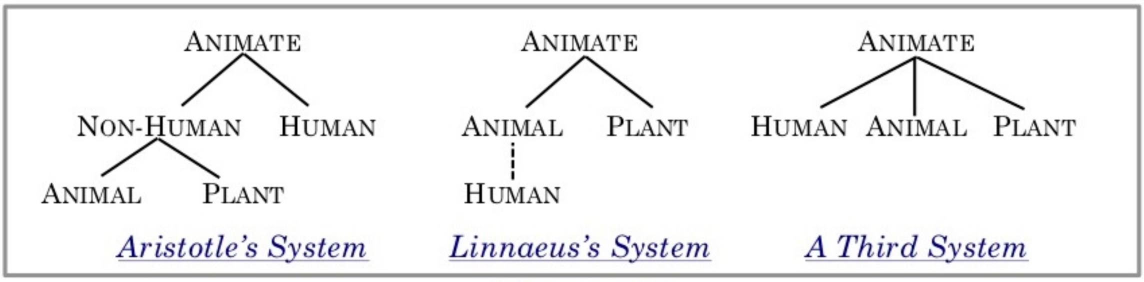


Figure 2-2

Which of these systems would you choose in order to build a theory of animate entities that maximises conclusions and minimises premises? State the reasons for your choice, based on the Principle of Logical Inheritance, and the need to minimise postulates.

[Note: This is not an easy pursuit! It is a long-term one, so take your time. We urge you to play around with these classificatory systems, whether alone or with your friends, and try to write up your thoughts on it. ☺]

2.4 An Example of a Derivation

We have seen examples of derivations using the combination of

(a) categories and sub-categories, and

(b) the principle of logical inheritance

to simplify the specification of the properties of entities. But it might be useful to take one more example.

Consider the following description of the category of humans:

2) Description of human beings:

a. Human beings have alimentary canals.

b. Human beings have vertebral columns.

c. Human beings have lungs.

Now, many other species have this combination of properties: chimpanzees, mice, cats, dogs, and elephants, to name a few. School science textbooks tell us that all these are mammals:

3) Description of mammals:

a. Mammals have alimentary canals.

b. Mammals have vertebral columns.

c. Mammals have lungs.

Given (3a-c), we can derive (2a-c) by simply adding a statement of sub-categorization to (2):

2) d. Human beings are a subcategory of mammals.

Also recall, from Ch1–(4), the **Principle of Logical Inheritance (PLI)**:

All the properties of a category are inherited by its subcategories.

We can now derive the statements in (2a-c) as follows:

Derivation of (2a-c):	
Human beings are a subcategory of mammals.	(2d)
Mammals have alimentary canals.	(3a)
Therefore, human beings have alimentary canals.	By PLI
Mammals have vertebral columns.	(2b)
Therefore, human beings have vertebral columns.	By PLI
Mammals have lungs.	(2.c)
Therefore human being have lungs.	By PLI

Given that (2a-c) are ***predictable*** — as shown by the above derivation, the only stipulation we need for humans is (2d). We can leave out the rest of the description of humans.

We now see that some of the properties of mammals specified in (3) are also shared by some non-mammals. For instance, birds, lizards, snakes, and fish also have vertebrae, and are subcategories of the category 'vertebrate'.

4) Description of vertebrates:
 a. Vertebrates have alimentary canals.
 b. Vertebrates have vertebral columns.

Given (4a, b), we can derive (3a, b) by simply adding a statement of sub-categorization to (3):

3) d. Mammals are a subcategory of vertebrates.

We can now derive the statements in (3a, b) as follows:

Derivation of (3a, b):	
Mammals are a subcategory of vertebrates.	(3d)
Vertebrates have alimentary canals.	(4a)
Therefore, mammals have alimentary canals.	By PLI
Vertebrates have vertebral columns.	(4b)
Therefore, mammals have vertebral columns.	By PLI

Is it possible to derive (3c) also by adding, "Vertebrates have lungs"? We leave that for you to find out. But given that (3a, b) are predictable, we can leave them out from the description of mammals, and stipulate only (3d).

Now, vertebrates are not the only life forms that have alimentary canals. Insects, worms, snakes, and so on also have alimentary canals. Is it possible to derive these properties from the properties of a more general category? We leave that also for you to figure out.

Also note that vertebrates are multicellular. And school textbooks tell us that vertebrates have eukaryotic cells. Is it possible to derive these properties from the properties of a more general category? That too, we leave to you to find out.

2.5 What We Learnt So Far And What Lies Ahead

In Chapter 1, we were introduced to the art and craft of theory construction by deducing *what is predictable* from *what is not predictable* in descriptions of geometric objects. This strategy called for categorising *premises* into the subcategories of *definitions* and *axioms,* and of *axioms* into the subcategories of *General Axioms* and *Discipline-Specific Axioms.*

In Chapter 2, we extended what we learnt in Chapter 1 to construct a theory that integrates theories of Parallelograms, Rectangles, and Squares. We also practiced using that methodology to construct a theory of birds and a theory of vertebrae.

When we are deciding how to categorise something, or what to treat as a *premise* and what to treat as a *conclusion*, one of the norms we rely on is that of *minimising premises* and *maximising the range of conclusions.* When we have to choose from among competing theories, we choose the option that most effectively minimises premises, and maximises predictions.

Going forward, in Chapter 3, we will integrate the theories of Triangles, Quadrilaterals, and Pentagons to construct a theory that would extend to all Polygons: Hexagons, Heptagons, Octagons, and so on. We will then proceed to Lines and Points in Chapter 4.

CHAPTER 3:
A THEORY OF POLYGONS

3.1 Looking Back

If you wish to become a water colour painter, you need a set of ***tools*** like pencils and brushes, and ***materials*** like water colour paints and sheets of water colour paper. You also need to learn a number of ***techniques***. (If you would like to watch a water colour painter in action, check out the 12-minute video at https://www.youtube.com/watch?v=H0R3_uGgzuM, "Simple Watercolour Landscape Painting Using Only One Brush!".)

Similarly, if you wish to learn to cook, you need tools like a stove, pans, spoons, knives, cutting boards, graters, grinders, and so on; and ***ingredients*** like vegetables, grains, oils, spices, etc. And you need to master certain techniques like boiling, steaming, frying, and baking. Similar remarks apply to learning how to dance, or sing, and to do carpentry, gardening, farming, healing, dentistry, surgery, and so on.

These are activities that you engage in with your body — eyes, ears, tongue, vocal organs, fingers, hands, arms, legs, and so on. Curriculum designers often use the term 'hands-on' skills and abilities and 'sensory-motor' skills to refer to what you need to learn in such domains.

In contrast, acquiring the capacity for mathematical, scientific, and philosophical, and other modes of inquiries involves '**minds-on**' activities, which call for thinking, reading, writing, listening, communicating, discussing, and debating. Like hands-on activities, these activities also need tools and techniques — not physical ones, but abstract mental tools and techniques. In textbooks and courses that teach you how to do research, they are often called ***methodological strategies***.

In Chapters 1 and 2, we dealt with some of the methodological strategies of ***theory construction***. These included:

> ***defining, categorising, abstracting, and integrating***.

These strategies are relevant in all domains of academic inquiry. Needless to say, you may also think of them as ***tools*** *or* ***techniques*** used for building knowledge.

As you can see, the items in this list of strategies are all verbs describing certain abilities. They also have corresponding noun forms. We call them ***CONCEPTS*** of academic knowledge and inquiry/research:

> ***definition, category, abstraction,*** and ***integration.***

In Chapter 1, without using the word *proof*, we unpacked the concept of ***PROOF*** in terms of ***premises, derivation,*** and ***conclusion*** (Fig. 1.5).

We made a distinction between two kinds of premises: ***axioms*** and ***definitions***. We saw the methodological strategy of ***axiomatising***. To axiomatise is to explicitly state the axioms and definitions of a proof, such that we can critically evaluate the merit of proofs (that is, find out if a proof is a good one). We looked at the concept of an ***axiomatic system*** — a system based on *axiomatising*.

In Section 1.6.3, we made a distinction between discipline-specific axioms of a particular field, and ***general axioms*** that are part of all academic fields. As you may have guessed, general axioms are trans-disciplinary.

We may make the same distinction among definitions. In physics, *force* is defined as *that which causes a change in the velocity of an inanimate entity*. But the definition of *force* as *that which causes change* is trans-disciplinary. It can be extended to the animate domain, including to humans.

We discussed the concepts and strategies of theory construction in the specific contexts of geometry and biology. However, these concepts and strategies are important for constructing theories in any domain of study, ranging from mathematics, and the physical-biological-human sciences, to the humanities. So we refer to them as ***trans-disciplinary*** concepts and strategies. We are using *trans-* in the sense of something that exists at an abstract level, cutting across fields, disciplines, and discipline groups.

Let us distinguish some of the levels of abstraction that are often referred to in the context of academic knowledge:

LEVELS OF ABSTRACTION

Inter-disciplinary: refers to the area of overlap between two disciplines. For example, biochemistry is interdisciplinary because the investigation of molecules is part of both biology and chemistry.

Multi-disciplinary: refers to questions/problems that call for the use of knowledge and strategies from multiple disciplines. For example, medicine is multi-disciplinary because to solve the problems of health, it uses knowledge from biology, psychology, chemistry, physics, ecology, sociology, and so on.

Trans-disciplinary: refers to questions, concepts, and inquiry strategies that are found across disciplines, and do not belong to any particular discipline. Take the example of ***CHANGE***. The concept of *change of location* (motion) and *change of velocity* (acceleration) are specific to physics, but the concept of *change* itself is part of biology (developmental changes, evolutionary changes); chemistry (chemical changes); sociology (social changes); geology (geological changes); and so on. Likewise, the concepts of skeletal structure, the structure of cells, and the structure of biomolecules are specific to biology. And the concept of the *structure of a poem*, the *structure of a dance*, and the *structure of a painting* are specific to their respective domains. However, the concept of ***STRUCTURE***, and the relation of ***IS COMPOSED OF*** as a central ingredient of structure, and even the very concept of ***CONCEPT*** are trans-disciplinary; they are not located in any specific discipline.

What do we mean when we say we 'know' something? How do we come to know something? How do we know if it is true? These questions come under the ***study of knowledge***. In philosophy, this is called ***epistemology*** .

Epistemology is the study of the nature of knowledge, ways of arriving at it, critically evaluating it, and proving claims to establish them as part of knowledge, including ***academic knowledge***. (If you are interested in finding out more about epistemology, watch the 6-minute video, "Philosophy - Epistemology: Introduction to Theory of Knowledge," at (https://www.youtube.com/watch?v=r_Y3utIeTPg&t=8s)))

ACADEMIC KNOWLEDGE may be thought of as the category of knowledge that comes under such things as mathematics, physics, chemistry, biology, psychology, sociology, economics, history, literary studies, and so on, and is transmitted through educational institutions (schools, colleges, universities, ...).

It is important to bear in mind that there are other forms of knowledge, such as ***experiential knowledge***, ***commonsense knowledge***, ***traditional knowledge***, and what is sometimes called 'wisdom'. This book is about academic knowledge, which is just one of the categories of knowledge, the one that is relevant for what we learn in educational institutions.

In this book, we will be concerned with the *epistemology of academic knowledge*. In particular, we will be concerned with the epistemology of mathematics, extending it briefly to biology. A comparison between mathematics and biology would prompt us to engage with important questions, like this one:

> What is the distinction between *proving something in mathematics* and *proving something in science?*

Now, in the context of academic knowledge, proving is the use of ***reasoning*** to convince someone that a knowledge claim is true. So let us take a detour into reasoning before we proceed to the concept of ***PROOF*** in the subsequent section.

3.2 Reasoning

What is reasoning? We may say that:

> REASONING is *the process by which we arrive at inferences.*

What does that mean? Suppose we are told that Anu is taller than Rafa. We are also told that Rafa is taller than Neel. From these two statements, we can infer a new statement: that Anu is taller than Neel. Similarly, if we are told that Neel's room is rectangular in shape, and that the length and breadth of his room are 15 feet and 10 feet respectively, we can infer that the area of his room is 150 square feet.

In Chapter 1 (section 1.5), we pointed out that in the study of reasoning, called logic, the inferences that we arrive at are called conclusions. The statements from which we derive the conclusions are called premises. And we refer to the steps of reasoning from the premises to the conclusion as derivation. We have used the term ***PDC Structure*** to refer to the structure of reasoning in terms of premises, derivation, and conclusion.

[Note: Our book, *Academic Knowledge and Inquiry across Disciplines*, has three chapters devoted to the study of reasoning. It may be a good idea to take a look at those chapters before proceeding to the next section.] **[Hyperlink book]**

3.3 The Concept of Proof

In Chapter 1, as stated above, we introduced the concepts of *premises*, *derivation*, and *conclusion*, which make up the PDC structure. A derivation shows that the conclusion logically follows from the premises. A few more examples of derivations would help us get a firmer grip on this concept. In these examples, each step in the derivation is legitimised by an earlier PREMISE (P) or a previous STEP (S).

Statement to be proved: Athena is a good cook.

Premises

P1. If Plato has a beard, then Aristotle likes books.
P2. If Aristotle likes books, then Athena is a good cook.
P3. Plato has a beard.

Derivation: Steps

S1:	If Plato has a beard, then Aristotle likes books.	(P1)
S2:	Plato has a beard.	(P3)
S3:	Therefore, Aristotle likes books.	(by S1, S2)
S4:	If Aristotle likes books, then Athena is a good cook.	(P2)
S5:	Therefore, Athena is a good cook.	(by S3, P2)

Conclusion

Athena is a good cook. (QED)

NOTE: QED is an abbreviation for the Latin expression
Quod Erat Demonstrandum,
which means:
'that which is to be demonstrated' (= to be proved).

Let us take another example:

Statement to be proved: Athena is taller than Apollo.

Premises

P1. Athena is taller than Xena.
P2. Xena is taller than Plato.
P3. Plato is taller than Apollo.

Derivation: Steps

S1:	Athena is taller than Xena.	(P1)
S2:	Xena is taller than Plato.	(P2)
S3:	Therefore, Athena is taller than Plato.	(by S1, S2)
S4:	Plato is taller than Apollo.	(P3)
S5:	Therefore, Athena is taller than Apollo.	(by S3, P3)

Conclusion

Athena is taller than Apollo. (QED)

In Ch1–(6) [read as: Chapter 1, item (6)], we derived the conclusion that RATs have three angles. Here is a look at the same proof, but stated in terms of premises, derivation, and conclusion.

Statement to be proved: RATs have three angles.

Premises

P1. A Triangle has three angles.
P2. A RAT is subcategory of Triangle.
P3. The properties of a category are inherited by their subcategories.

Derivation: Steps

S1:	The properties of a category are inherited by its subcategories.	(P3)
S2:	A RAT is subcategory of Triangles.	(P2)
S3:	The properties of Triangles are inherited by RATs.	(by P2-P3)
S4:	A Triangle has three angles.	(P1)
S5:	Therefore, RATs have three angles.	(by S3-P1)

Conclusion

RATs have three angles. (QED)

Notice that the discipline-specific axiom of subcategorisation in (P2) and the trans-disciplinary axiom that we called ‘the Principle of Logical Inheritance’ (PLI) relevant for (P3) are both central to this proof.

Proofs that appeal to subcategorisation and the accompanying inheritance of properties are essential for integrating the theories of RATs, ETs, and Triangles into a single theory. This point can be generalised as:

> Categorisation and subcategorisation play an important role in the integration of academic knowledge.

We will have more occasions to use this strategy in the integration of other theories.

To take another example of a proof, this time from number theory, consider Euclid's Proof of the infinity of prime numbers. If you haven't come across this proof, it may be useful to watch this YouTube video, which presents Euclid's Proof reasonably well:

"Euclid's Proof that there an Infinite Number of Prime Numbers"
at https://www.youtube.com/watch?v=OxGRl0phiB4

Exercise 1

Can you think of a way to derive the properties of Triangles and Quadrilaterals from the properties of Polygons, in order to avoid their duplication in each category?

3.4 Properties and Relations

In Section 1.7, we distinguished between ***descriptions*** and ***theories***. A description of an entity is a body of information about that entity. In chemistry, for example, the description of an entity is a set of structural or behavioural properties of that entity. Let us take water, for example, which has the following properties:

Water:
~ It is a liquid.
~ It solidifies to ice when frozen.
~ Its freezing point is 0ºC.
~ It becomes steam when boiled.
~ Its boiling point is 100ºC.
~ It is not inflammable.
~ Sugar dissolves in it.
and so on.

Each of these statements forms part of a description of an entity, that is, a *description fragment*. When we compare different entities, we find differences in their properties, and can state them as contrasting description fragments, for example:

~ copper is malleable, but glass is brittle.
~ petrol is inflammable, but water is not.
~ honey is sweet, while the juice of a lime is sour.
and so on.

Some properties of entities can be derived from the categories they belong to. Thus, given that copper is a subcategory of the category 'metal', we can derive the statement that copper is malleable from the more general statement that metals are malleable. Similarly, since butterflies are insects, we can derive the property of butterflies having six legs from a more general statement that insects have six legs. These examples of deriving specific statements from more general ones take for granted a system of categories and subcategories that we are familiar with: copper as a subcategory of metal, and butterfly as a subcategory of insect. But in other cases, we may have to construct, or modify, the system of categories needed for generalisation.

When we are looking at the properties of entities that we want our theory to predict (to derive as a conclusion or a theorem), it is useful to think of not only ***properties***, but also ***relations***. Here are a few examples to illustrate the difference:

Property	*Relation*
Zeno IS OLD.	Zeno IS OLDER THAN Plato.
Zeno IS YOUNG.	Zeno IS YOUNGER THAN Plato.
Zeno IS MARRIED.	Zeno IS MARRIED TO Athena.
Zeno IS A TEACHER.	Zeno TEACHES Plato.

As in the case of axioms and definitions, properties and relations can belong to a specific field, a discipline, a discipline group, or be trans-disciplinary.

We are interested in both discipline-specific and trans-disciplinary properties and relations. Trans-disciplinary relations that appear in theories across disciplines, under varying terminologies, include:

CONCEPT	TERMS (VARIANT EQUIVALENTS)
Subcategorisation	*x* is A SUBCATEGORY OF *y*
Compositionality	*x* is COMPOSED OF *y, z, …* Variants: *x* is MADE UP OF *y, z, …* *x* is A CONSTITUENT OF *y, z, …* *x* is DECOMPOSABLE INTO *y, z, …*
Ordering	*x* is ORDERED PRIOR TO *y* Variants: *x* is RANKED HIGHER THAN *y* *x* PRECEDES *y*
Logical Consequence	*x* is a LOGICAL CONSEQUENCE OF *y*
Logical Contradiction	*x* LOGICALLY CONTRADICTS *y*
Equality	*x* IS EQUAL TO *y* Variants: *x* IS EQUIVALENT TO *y* *x* IS AN ANALOGUE OF *y* *x* IS A HOMOLOGUE OF *y* …
Correlation	*x* CORRELATES WITH *y*
Causation	*x* CAUSES *y*
Instantiation	*x* is AN INSTANCE OF *y* Variants: *x* is A MEMBER OF set/category *y* *x* is AN EXAMPLE OF *y* *x* is A SAMPLE OF *y* …
Negation	*x* is THE NEGATION OF *y* Variants: *x* is THE OPPOSITE OF *y*

It would be worth emphasising that in this table, we are giving different but equivalent terminologies used in different fields, to help learners become aware of corresponding terms, and start seeing trans-disciplinarity in spite of seeming differences in terminologies.

When we take a number of description fragments such as those illustrated above, generalise them, and place the generalisations in a PDC structure, we get a *theory*. A theory is subject to the requirement that *the smallest possible number of premises should yield the widest range of conclusions.*

We have discussed subcategorisation in Sections 1.4, 1.5, 2.2, and 2.3. In conjunction with the axiom of the logical inheritance of properties (Ch1 (4)), the use of the subcategory relation allows us to deduce seemingly arbitrary properties of a category from its mother category. This facilitates the integration of specialised theories into general theories.

3.5 Compositionality

We have seen compositionality appear in geometry in statements like:

A triangle *is composed of* exactly three vertices, and exactly three straight lines connecting them.

A quadrilateral *is composed of* exactly four vertices, and exactly four straight lines connecting them.

A pentagon i*s composed of* exactly five vertices, and exactly five straight lines connecting them.

[Note: The term *vertices* is the plural form of the singular *vertex*.]

COMPOSITIONALITY is a ***part-whole relation.*** In statements of the form "*x is composed of y, z…* ," *x* is the whole, and the rest are parts of that whole. Thus, we can say:

A human body is composed of organs;
An organ is composed of tissues;
A tissue is composed of cells;
A cell is composed of molecules;
A molecule is composed of atoms;
An atom is composed of particles.

As in geometry and in biology, it would be useful to explore compositionality in domains like arithmetic. For instance, consider the following statements:

a. 10 is the **sum of** 6 and 4.
b. 10 is a **multiple of** 5.
c. 10 is the **product of** 2 and 5.
d. 10 is **divisible by** 2.

We may imagine these concepts (in bold face) in terms of inputs and outputs. For example, (a) can be written as: "when we add 6 and 4, we get 10"; and (d) can be written as: "when we divide 10 by 2, we get a whole number."

We may also think of these statements as relations. In this view, "X is the sum of Y and Z" is a relation between X on the one hand, and Y and Z on the other. In (a), X is 10, Y is 6, and Z is 4. "Is divisible by" in (d) is a relation between 10 and 2.

It would be useful to pause and reflect on all the examples of relations we have come across.

3.6 Towards an Integrated Theory of Polygons

In Chapters 1 and 2, we introduced the idea of categorising and subcategorising as a tool to integrate two or more special theories into a single general theory. For example, we integrated three otherwise unrelated theories — of RATs, ETs, and Triangles — into a single theory of Triangles, by postulating that RATs and ETs are subcategories of Triangles (Sections 1.4; 1.5).

Similarly, we integrated the otherwise unrelated theories of Squares, Rectangles, Rhombuses, and Parallelograms into a single theory of Quadrilaterals. The following axioms allow us to do this:

Squares are a subcategory of Rectangles.
Rectangles are a subcategory of Parallelograms.
Squares are a subcategory of Rhombuses.
Rhombuses are a subcategory of Parallelograms.
Parallelograms are a subcategory of Quadrilaterals.

To this, we may add:

Triangles, Quadrilaterals, Pentagons, Hexagons,…
are subcategories of Polygons.

3.7 Categorisation and Compositionality in Conjectures

In mathematics, the term ***conjecture*** refers to a statement that we think is true, but have not proved it yet. When a conjecture is proved, it is called a ***theorem***.

Let us try some thought experiments to see the roles of categorisation and compositionality in the process of proving conjectures to establish them as theorems.

In your mind, construct a Square. In that Square, draw diagonals. How many diagonals can you draw? No more than two, right?

Now imagine a Rectangle, and draw diagonals in it. How many can you draw? Again, no more than two, right?

Do the same with a Rhombus. How many diagonals? No more than two, of course.

Let us generalise, first to a Parallelogram, and then to any Quadrilateral.

For this, let us begin with a couple of conjectures, one that is most general, and one that is the most specific:

Conjecture 1: A geometric object has no more than two diagonals.

Conjecture 2: A square has no more than two diagonals.

Exercise 2:

TASK 1: For Conjecture 1, and Conjecture2:
If you think it is true, try to prove it.
If you think it is false, try to disprove it.

To do this, you need to first define the concept of diagonal clearly and precisely.

TASK 2: Ask yourself if there are geometric objects with three or more diagonals. For this, imagine geometric objects that allow three or more diagonals.

It is absolutely important that you complete these tasks before you proceed to the next exercise. This is an opportunity to learn experientially what a *minds-on* activity is (rather than a *hands-on* one).

Exercise 3

TASK 1: Using what you have learnt in this chapter, especially by going through Exercise 1, come up with conjectures about the categories of pentagons, hexagons, and septagons.

Formulate a conjecture about diagonals in polygons in general.

TASK 2: Now try to formulate a conjecture about diagonals in geometric objects in general. Does this instruction trigger an unease in your mind? Why? Think of a possible reason, and state it in writing, clearly and precisely. [This is not an easy task.]

From the geometry you learnt in school, you are already familiar with the terminology of Right-Angled Triangle, Equilateral Triangle, Equiangular Triangle, Isosceles Triangle, Square, Rectangle, Parallelogram, Rhombus, Quadrilateral, Pentagon, Hexagon, … and Polygon, and also the idea of Point, Straight Line, Curved Line, Vertex, Angle, Side, and Diagonal. The tasks in Exercises 1 and 2 were meant to give you a feel for what it is like to formulate conjectures, and prove them, drawing on what you have already learnt.

When we do the thought experiment of drawing a diagonal in a square in our mind, we see in our mind's eye, without having to draw it on a piece of paper, that a square with one diagonal is composed of two triangles. Again, without drawing it on paper, we can see that the two triangles are right-angled isosceles triangles.

Let us write what we have discovered through the thought experiments:

Conjecture 3: Any square can be divided into two right-angled isosceles triangles.

Conjecture 4: Any pair of right-angled isosceles triangles of the same size can be joined to form a square.

The statements that you arrived at in Exercises 1–3 involve the relations between categories, and the relation of compositionality. The combination of these two kinds of relations is integral to the concept of ***structure***. This concept is of value not only in geometry, but in all domains of academic inquiry.

In other words, structure as described above is a transdisciplinary concept. You could view what we are doing in these exercises as beginning to construct ***a theory of the structure*** of polygons.

[We discussed the *Structure of Theories* in Chapter 1. We are now talking about the *Theory of Structur*e.]

The truth of the statements in the theory you have constructed in Exercises 1–3 may be 'obvious' to you. But we must remind you that it is based on a sample of geometric objects. For example, the statement, "A square has no nore than two diagonals," is based on thought experiments with a sample of squares you can see in your mind's eye. You cannot do thought experiments on every square in an infinite population of squares. So, even though they may be 'obvious' to you, you have not yet established them as true.

In other words, these statements are conjectures. They have not yet been proved, so they are not theorems yet. They are, however, ***plausible*** *conjectures*. And here is why. Their truth may be obvious to us, but what if they are *not* obvious to the others in our research community? What if they demand proofs before they accept the propositions as true?

> ***PLAUSIBLE CONJECTURE***
>
> A plausible conjecture is one that is most likely to be true, but has not been proved yet. A famous conjecture, called Goldbach's conjecture, says that every even integer greater than 2 is equal to the sum of two prime numbers. It has not been proved yet, but all mathematicians are convinced that it is true. This is a plausible conjecture, but not yet a theorem.
>
> (https://www.britannica.com/science/Goldbach-conjecture)

We could think of the outcome of Exercise 3 as an example of *simulated research*. It is *research* because, to do these exercises, one has to go through the process of inquiry. Inquiry forms the core of research. And it is *simulated* because the theory that one comes up with is already known to the community of mathematicians, so it is not a new contribution to the field. (To qualify as research, the outcome of the process must make a contribution to the existing body of knowledge.)

Now, saying that it is only *simulated* research and not actual research should not dampen anyone's enthusiasm. If you continue along the journey in this book, and practice several times what you have learnt, taking new research questions each time, you will hone your abilities and be ready to make contributions to the existing body of knowledge in your chosen field of study.

Exercise 4

The theory that we have constructed so far is on triangles and squares.

TASK: Generalise it to construct a theory of triangles and quadrilaterals.

Exercise 5

Triangles and quadrilaterals are polygons.

TASK: Construct an integrated theory of polygons, and present it as a written document, with as much clarity, precision, and attention to detail as you can. You can do this as an individual project, or a joint project with others.

Our practice of the methodological strategies of theory construction in Exercises 1–5 has been restricted to the terrain of geometry. For further practice, and for you to get a first-hand experience of seeing what is transdisciplinary about the methodological tools we have used, let us move from geometry to biology.

Exercise 6

TASK: Based on the biology exercises in Chapter 2 (Exercises 2–4), construct an integrated theory of the structure of the categories and of organs discussed in the exercises, and write down your findings.

This is not an exercise that can be done in a few hours, days, or even weeks. It could take you a month or two, perhaps even more. You may also rope in collaborators to do this project.

3.8 Summing up

In Chapter 1, we demonstrated how we can construct an integrated theory of triangles, and Chapter 2 showed how we can construct an integrated theory of quadrilaterals. In Sections 3.1 – 3.4, we demonstrated how we begin to construct an integrated theory of polygons.

Some of the transdisciplinary tools (strategies and concepts) that we have used so far for constructing theories in the terrain of geometry include:

Strategies:	categorising and subcategorising; defining; decomposing; generalising, integrating; proving
Concepts:	structure; compositionality; categories and subcategories; premises (axioms and definitions), derivation, and conclusion; conjectures and theorems; predictions; proof

We have also seen the trans-disciplinary nature of the tools and concepts of inquiry by using them to construct and evaluate theories in biology.

Chapter 4:
A Theory of Lines and Points

4.1 Lines without Breadth and Points without Size

As a child, you must have first learnt about the concepts of lines and points by making marks with a pencil or a crayon on a piece of paper. For many of us, this is the understanding of 'line' and 'point' that stays with us.

Figure 4-1

Figure 4-2

In textbooks, you would have seen that the marks on paper in Fig. 4-1 are called ***lines***, and the ones in Fig. 4-2 are called ***points***. Later, we learn statements like these:

> *Lines have length, but no breadth or thickness.*
> *Points have no length, breadth, or thickness: they have no size.*

This doesn't makes sense, right? We can see the marks in Fig. 4-1, and they do have breadth, however small. And the marks in Fig. 4-2 have some area, no matter how small. Our experience contradicts what we learn in school. However, we have neither the conceptual clarity nor the vocabulary to voice the discomfort. So we often bury such queries and accept what we are told.

But this is a problem. Diagrams like the ones in Figs. 4-1 and 4-2 can hinder a conceptual understanding of geometry. How can we avoid this dissonance?

Here is an option. Suppose we define lines and points as follows:

LINE (DEF): A line is the boundary (edge) of a region.
POINT (DEF): A point is an intersection between two lines.

Figure 4-3

Figure 4-4

To elaborate, consider Figs. 4-3 and 4-4.

How many REGIONS does Fig. 4-3 have? It has three regions: the purple region, with an orange region and a green region inside it. In geometry, we call the purple region the *space*, and the orange and green regions would be *objects* in that space.

How many LINES does Fig. 4-3 have? Take the green object. Let us say that it has two edges, a straight one and a curved one. Let us interpret the edges as lines. How about the orange rectangle? It has four straight edges. So, a semicircle is made of two lines, and a rectangle, of four lines.

Suppose the objects in Fig. 4-3 are also coloured purple. Now, we cannot see the objects. How can we know that they are there? For this, we would have to colour the edges differently, as in Fig. 4-4. If we ignore the notation of colour, Figs 4-3 and 4-4 are identical; the concepts they express are also identical.

Now think about this. The edges in Fig. 4-3, do not have any breadth. But in Fig. 4-4, the white marks (lines) that indicate the edges seem to have breadth. However, this is only because the white marks on the purple *represent* the geometric objects; they are not lines. They have non-zero breadth,only because otherwise we would not be able to see them.

[Note: Remember that we are using colours here only to gain a better understanding of the concept of line. The concept of colour is *not* relevant in geometry.]

Is the above discussion of lines clear to you? If not, try this. Take the black mark on gray that represents a straight line in Fig. 4-1. Imagine it getting thicker and thicker. At some point, it won't look like a line, but like a black rectangle. Now imagine the same image in your mind getting thinner and thinner, till it looks like the line in Fig. 4-1. Continue making the image thinner, till it is 0.0001 mm. You can't see it any longer, right? Make it even thinner. It has practically no breadth. But it is still a line, isn't it? Now can you imagine a line with zero breadth?

Try doing the same thing with the biggest dot in Fig. 4-2. Make it bigger and bigger till it looks like a large circle. Now make it smaller and smaller till its size is 0.0001 mm. Can you see it? No, it's practically invisible! So can you imagine a point with no area?

When Euclid talked about lines with no breadth and points with no size, he was talking about such invisible lines and points.

We tend to take for granted that the concept of line is something we know. But in an academic discussion, we cannot talk about a concept without first clarifying it, right?

4.2 Points in a Line

In the discussion above, we outlined Euclid's position that lines have zero breadth, and points have zero magnitude/size. But what if we conceptualise points as geometric objects with extremely small size, like dots made with an extremely well-sharpened pencil? Now conceptualise a line as an object made of a number of points. We can then define the length of a line as:

> LENGTH (DEF): The length of a line is the number of points it contains.

Within this conception — let us call it a discrete geometry — given finite lines A and B, if A has more points than B, A is longer than B. But in Euclidean geometry, every finite line has infinitely many points. So the statement that A has more points than B does not make sense.

What if we were to set up the following axiom in discrete geometry?

Axiom: Every line of finite length has a finite number of points.

Notice that since Euclid assumes that points have zero length and zero breadth, adding points together to make a line still gives us zero length.

Also, in Euclidean geometry, no matter how close two points are, they cannot be adjacent (that is, next to each other), because there is always at least one point between them. To see what this means, let us look at numbers.

In the world of integers, numbers 3 and 4 are adjacent because there is no integer between them. Integers 458 and 459 are adjacent too, because there is no integer between them. In contrast, 3 and 7 are not adjacent, nor are 458 and 464.

In the world of rational numbers, however, 3 and 4 are not adjacent. For example, between 3 and 4, we have 3.2, 3.3, … 3.6, … 3.9, and so on. How about 3.2 and 3.3? Are they adjacent? No, because we have 3.23 between them. What about 3.24 and 3.25? They are not adjacent either, because 3.242 lies between them.

In short, points in Euclidean geometry are like rational numbers, not like integers: they cannot be adjacent. In the discrete geometry that we are setting up now, we are treating points like integers, where two points can indeed be adjacent.

We now have two theories of geometry: the geometry of Euclid's *Elements* (where points are like rational numbers), and the discrete geometry that we have just set up (where points are like integers).

We might now ask: Do these two theories form *distinct* theories of geometry? Do they yield theorems that logically contradict one another?

Let us take a look.

Take the concept of *bisection*. Theorem 10 in Euclid's *Elements* says:

Every straight line segment is bisectable.

This means that every straight line segment can be cut into two lines that are congruent (of the same length).

This is taken to be true of Euclidean geometry. But is it true in the theory of discrete geometry?

Exercise 1

TASK: Define BISECTION.

Then figure out if the following conjecture is true:

Conjecture: There exist lines which are not bisectable.

If you think it is true, prove it.

Anytime we encounter two theories of something, we need to find out if they are two distinct theories, or the same theory that may look different.

Exercise 1 is important because it shows a way of engaging with that issue. If the two theories yield distinct predictions, the theories are distinct. But if they yield exactly the same set of predictions, they are not distinct theories: they just look different because they use different words and diagrams.

4.3 Parallel Lines

Intuitively, we judge the lines in Fig. 4-5 to be "parallel", and those in Fig. 4-6 to be not parallel.

Figure 4-5 ***Figure 4-6***

The basis for this intuition is that we think of parallel lines as having certain properties, and all these properties are satisfied by the lines in Fig. 4-5, but none of them by those in Fig. 4-6.

Table 1

	Property	Fig. 4-5	Fig. 4-6
a.	Equi-distance: The distance between the two lines is *invariant* (remains the same) even when they are extended indefinitely on both sides.	**YES**	**NO**
b.	Non-intersectability: The lines never intersect even when extended indefinitely on both sides.	**YES**	**NO**
c.	Perpendicular intersection: It is possible to have an intersecting straight line perpendicular to both the lines.	**YES**	**NO**
d.	Conservation of the sum of internal angles: If a straight line intersects the two lines, the sum of internal angles on the same side is two right angles.	**YES**	**NO**

In Euclidean geometry, the properties in Table 1 converge: any pair of straight lines that has one of these properties also has all the others. Hence, we may take *any* of these properties as defining the concept of parallel, and derive the remaining properties as theorems.

In non-Euclidean geometries, it is possible for a pair of lines to have one or more of these properties but not necessarily all of them. Let's explore this a little further.

The term "parallel" is conventionally used only in the context of straight lines. However, unless the definition specifies that the term is restricted to straight lines, we can take them to hold good for curved lines as well.

For example, take the following pairs of curved lines:

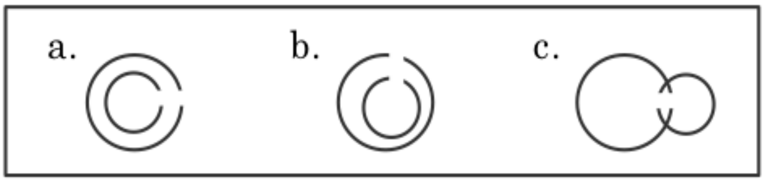


Figure 4-7

All the pairs of lines in Fig. 4-7 are circular, and therefore, curved.

In Fig. 4-7a, the lines are concentric. Therefore:

~ they are equidistant;

~ they cannot intersect;

and provided we define *perpendicular* and *angle* in a way that they apply to curved lines:

~ a straight line drawn through the centre would be perpendicular to both lines; and

~ given any straight line intersecting the two lines, the sum of the internal angles on the same side would be two right angles.

Thus, Fig. 4-7a exhibits all the properties in Table 1. Unless the definition limits the concept of parallel to straight lines, the lines in Fig. 4-7a would count as parallel.

Now, when we use the word 'parallel', we often take it for granted that we are talking about straight lines. The lines in Fig. 4-7 are not straight lines; they are curved.

If we do not specify straightness of the lines for the concept of 'parallel', then the following definition should work:

Two lines are parallel if they never intersect.

By this definition, the lines in Fig. 4-7a and 4-7b are parallel. If we wish to judge the lines in these figures as not parallel, we need to say something along the following lines:

Two lines are parallel if they are both straight and never intersect.

Two straight lines are parallel if they never intersect.

How about Figs. 4-7b and 4-7c? Using what is given in Table 2, we can work out how these two figures relate to the properties of 'parallel-ness'.

Table 2

	Property	Fig. 4-7a	Fig. 4-7b	Fig. 4-7c
a.	Equi-distance	YES	NO	NO
b.	Non-intersectability	YES	YES	NO
c.	Perpendicular intersection	YES	NO	NO
d.	Conservation of sum of internal angles	YES	NO	NO

Whether or not the lines in Fig. 4-7b are parallel depends on how we define the term. If we take non-intersectability as its defining property, then the lines in the figure are parallel, but not otherwise.

How about the lines in Fig. 4-7c? They are not parallel no matter how we choose to define the concept of 'parallel'.

Exercise 2

Here are two definitions of parallel lines:

Definition 1: Two lines A and B are parallel iff there exists a line C perpendicular to both.

Definition 2: Two lines A and B are parallel iff they are equidistant.

TASK: Draw figures to demonstrate that both these definitions are flawed. Then revise the definitions in such a way that they are not flawed.

HINT: To show that these definitions are flawed, you would need to show that lines that you do not consider to be parallel are parallel according to the definition.

In our discussion of parallel lines, the concepts of angle and of straight lines vs. curved lines turned out to be crucial. We will explore curved lines further in Chapter 5. But in the meantime, it would be good for you to try to define *straight line*, *angle*, and *vertex*; and discuss your ideas with others who are interested.

4.4 Summing up

In the chapters so far, we have used the methodological strategies of *axiomatic inquiry in theory construction*. Axiomatic inquiry uses axiomatic systems. An axiomatic system consists of premises (axioms and definitions), derivations, and conclusions. The derivations call for a deductive system, i.e., one in which conclusions are derived through deductive logic. In axiomatic inquiry, the truth of a conclusion is proved by deducing it from the premises.

Mathematical theories are axiomatic systems. Theories outside of mathematics, such as scientific theories, also have axiomatic components, and they employ axiomatic systems. But in addition, they have an observational component. Scientific inquiry, for instance, employs an axiomatic system to deduce predictions, but the correctness of predictions is not established by demonstrating that they are deduced from axioms and definitions. Rather, scientific inquiry demands that the correctness of predications be established by showing that they match the observational generalisations. The norm of correctness in terms of observations does not hold for mathematical theories, since they are about logically possible imagined worlds created by us, and not about the real world.

We can identify two important properties of the axiomatic mode of inquiry that we learnt in Chapter 4:

A. A question about something cannot be answered unless that something is defined clearly and precisely.

B. If we change an axiom or a definition, it is most likely that the conclusions will also change.

In the chapters that follow, we will explore the function of axiomatic systems in scientific theories.

CHAPTER 5:
A THEORY OF CIRCLES

5.1 Circles: Regions vs. Boundaries

Let us go back to the concepts of regions and boundaries we saw earlier, in Chapter 4 (Section 4.1). Consider the two 'circles' in Figs. 5-1 and 5-2:

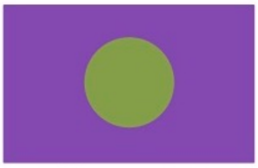


Figure 5-1

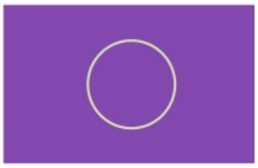


Figure 5-2

In Fig. 5-1, a purple rectangular region has a green circular region inside it. The green region has a boundary, which is also circular. In this figure, the boundary is an edge; it is a transition from one colour to another.

The purple rectangular region in Fig. 5-2 also has a circular region inside it, but here, this region is also purple, with a circular boundary in the form of a white line.

These two figures may help us to conceptualise 'circle' in two different ways:

Circle (CONCEPT 1): A circular region

Circle (CONCEPT 2): The boundary of a circular region

Euclid defines *Circle* as follows:

Circle (DEF 1): "A circle is a plane figure bounded by one curved line, and such that all straight lines drawn from a certain point within it to the bounding line, are equal."
Euclid, *Elements*, Book I[1]:4

Notice that Euclid's definition is that of 'circle' as a region (CONCEPT 1), not a boundary (CONCEPT 2).

A definition of 'circle' as a boundary (CONCEPT 2) would be:

Circle (DEF 2): A circle is a curved line which encloses a point from which all straight lines to any point on the curved line are equal.

Now, if we think of a circle-as-a-boundary as a trajectory of a point, we can define it as follows:

Circle (DEF-3): A circle is the trajectory of a point in motion such that there is a point from which all straight lines to the point in motion are equal.

In other words, we can refer to a circle as defined by CONCEPT 1 as a 'CIRCULAR DISK', and a circle as defined by CONCEPT 2 as a 'CIRCULAR LINE'.

[It may be useful to look up the Wikipedia entry on 'disk' at: https://en.wikipedia.org/wiki/Disk_(mathematics).]

Whether we go by CONCEPT 1 or CONCEPT 2, there are properties that are common to both:

(i) Every circle has a central point, called the CENTRE of the circle.

(ii) The boundary is the CIRCUMFERENCE.

(iii) The straight line from the centre to any point on the circumference is the RADIUS.

(iv) A straight line from any point on the circumference to a point on the opposite side, passing through the centre, is the DIAMETER.

Here is a simple exercise for you to try.

Exercise 1

TASK: Prove that the length of the diameter of a circle is twice the length of its radius.

To think about: Do circles as boundaries have area?

Do circles as regions have length?

As we saw in Chapter 4, we cannot answer these questions until we define the concepts of *region* and *boundary*. So, following Euclid's strategy, let us state our axioms:

Axiom 1: Regions have area.

Axiom 2: Boundaries of regions are lines that have length, but no breadth or area.

This means that a circle as a circular line (CONCEPT 2) has length but no area; and a circle as a circular disk (CONCEPT 1) has area, but no length.

Of course, we could have chosen to say that a circular line has both length and breadth. That is what we did in Section 4.2, where we chose to view lines as geometric objects that have breadth.

Here, we are taking a different track, and staying within Euclidean geometry.

Exercise 2

TASK: Keeping in mind the discussion about the two ways of defining lines in terms of boundaries (Chapter 4), propose a definition for 'circular line', such that:

(a) circle is a subcategory of the category 'circular line',

and then propose another definition for 'circular line', such that:

(b) circle is distinct from circular line.

Instead of defining length or clarifying the concept of length through axioms, Euclid set up the concept of ***congruence*** to replace the notion of 'same length' (or equal length). In this system:

> Two lines A and B are congruent iff one of them can be placed on top of the other such that they coincide exactly.

This should not be difficult to follow.

> In your mind, draw a straight line A, and another straight line B.
>
> Move one of them and place it on top of the other.
>
> If the end points of the two lines fit exactly, the lines are congruent.
>
> If they are congruent, one of the lines is inside the other, and is shorter.

Using the congruence test, is it possible to determine whether a straight line and a wavy line have the same length, or if one of them is shorter. How about a wavy line and a semi-circular line?

We cannot answer that question, because we have not defined or clarified length in such a way that we can compare lengths of lines with distinct shapes.

5.2 Circle Theorems

Do you remember the so-called 'circle theorems' from your geometry classes? They are about the correlation between circles and polygons. Many of them are correlations between circles and triangles. Here is an example:

> A triangle in which one of the sides is the diameter of a circle,
> and every vertex is on the circle, is a right-angled triangle.

We will state a few of them as conjectures. You could try to prove them to establish them as theorems. [IMPORTANT: Resist the temptation to do an Internet search for a proof that someone else has written. ☺]

Let us begin with the idea of circumscription and inscription.

Conjecture 1: For every triangle, there exists exactly one circle that circumscribes it.

Conjecture 2: For every triangle, there exists exactly one circle that is inscribed in it.

As we know very well by now, we can neither prove nor disprove these conjectures without defining *circumscription* and *inscription*. Let us try these definitions:

Circumscription (DEF): A circle circumscribes a triangle iff every vertex of the triangle is on the circle.

Inscription (DEF): A triangle inscribes a circle iff the circle touches every side of the triangle.

In both cases, the outer object Suppose we use the two words as inverses of each other, and hold that:

Given two geometric objects A and B, A circumscribes B
iff B inscribes A.

If we accept this view, we do not need two different concepts. We can use the word circumscribe in such a way that 'Y is inscribed in X' means the same as 'X circumscribes Y'. To illustrate, consider these diagrams:

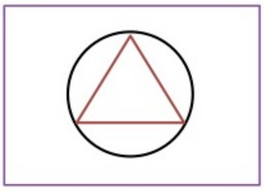


Figure 5-3

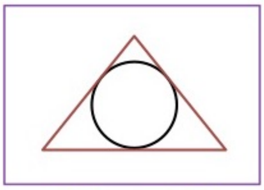


Figure 5-4

To be clear about what we mean by *circumscribe* and *inscribe*, we may describe the relation between the circle and the triangle in these two figures as follows:

~ In Fig. 5-3, the circle circumscribes the triangle.
(= The triangle is inscribed in the circle.)
~ In Fig. 5-4, the triangle circumscribes the circle.
(= The circle is inscribed in the triangle.)

This way of thinking about circumscription and inscription may be different from what you have learnt in school. But here is how Wikipedia describes them:

> "In geometry, an inscribed planar shape or solid is one that is enclosed by and "fits snugly" inside another geometric shape or solid. To say that "figure F is inscribed in figure G" means precisely the same thing as "figure G is circumscribed about figure F"."
> (https://en.wikipedia.org/wiki/Inscribed_figure)

If we adopt the Wikipedia conceptualisation, how do we define 'circumscribe'?

Exercise 4

TASK 1: Complete the following definition:

Geometric object X circumscribes geometric object Y iff …

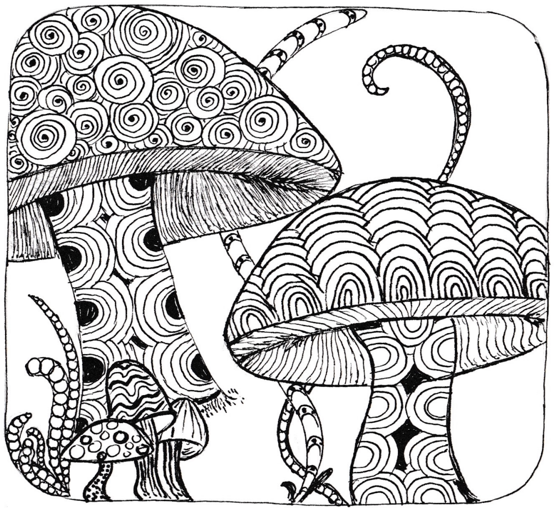
Using that definition, prove or disprove Conjectures 1 and 2.

TASK 2: Take a look at the eight circle theorems at
http://www.timdevereux.co.uk/maths/geompages/8theorem.php

For your own practice, treat these theorems as conjectures and try to prove as many of them as you can. It is okay to draw on your memory of the proofs that you are familiar with from school, but you must state them in terms of the premises-derivation-conclusion (PDC) structure illustrated in the previous Chapters.

Again, resist the temptation to do an Internet search to find existing proofs, because that would rob you of a learning opportunity. To develop the capacity to come up with and write proofs with clarity and precision, what matters is *the effort* that you put into this task (either individually or in groups), and not whether you are *successful* in arriving at a proof, or whether an expert judges it to be valid.

Now, the idea of 'validity of derivations' requires a bit of elucidation. See the box below.



The Concept of Validity

Suppose we are told that the following premises are true:

P1. Athena is taller than Apollo.
P2. Apollo is taller than Zeno.
P3. Plato is taller than Socrates.
P4. Socrates is taller than Aristotle.

We are now asked: Who is taller, Athena or Aristotle?

Are we in a position to answer the question? No, because one crucial premise needed for deriving an answer is missing. P1 to P4 allow us to infer that Athena is taller than Zeno, and that Plato is taller than Aristotle. But if someone were to conclude that Athena is taller than Aristotle, we would point out that the derivation is flawed, because there is no premise that connects Zeno and Plato. For this, we need to add:

P5. Zeno is taller than Plato.

We refer to a derivation as ***valid*** if it is without flaws, ***invalid*** if the derivation is flawed. The derivation of the conclusion that Athena is taller than Aristotle is valid given the premises in 1-5. But it is invalid if any one of those premises is missing.

Let us take a few more examples. We will consider Premise-Conclusion sets (PC), without necessarily spelling out the steps of the derivation. We will also take it that if the derivation is valid, then the conclusion is legitimate.

Consider the following example of a premise-conclusion set (PC1):

PC1: P1 The length of this rectangle is 12 cms.
C Therefore, its area is 120 sq. cms.

You would agree that given only P1 in PC1, the conclusion is not legitimate, even though it is true. This is because a premise about the breadth of the rectangle, which is required for arriving at the conclusion, is missing. Let us add that premise:

PC2: P1 The length of this rectangle is 12 cms.
P2 Its breadth is 10 cms.
C Therefore, its area is 120 sq. cms.

Now, we might think that PC2 is legitimate, but there is still something missing. To figure out what is missing, compare PC2 with PC3:

PC3: P1 The length of this parallelogram is 12 cms.
P2 Its breadth is 10 cms.
C Therefore, its area is 120 sq. cms.

You would agree that this conclusion is not legitimate. What makes the conclusion in PC2 legitimate, but not the one in PC3?

The reason is simple. In PC2, there is something that we happen to have learnt, perhaps in school. We are taking for granted that it is true, but have not stated it explicitly. Let us state it:

The area of a rectangle is its length multiplied by its breadth.

To make our conclusion in PC2 legitimate, let us add this premise to the set:

PC4: P1 The length of this rectangle is 12 cms.
P2 Its breadth is 10 cms.
P3 The area of a rectangle is its length multiplied by its breadth.
C Therefore, its area is 120 sq. cms.

In contrast, the premise about the area of a parallelogram that we take as true is:

The area of a parallelogram is its base multiplied by height.

To make our conclusion in PC3 legitimate, we add this premise to the set relating to parallelograms:

PC5: P1 The length of this parallelogram is 12 cms.
P2 Its breadth is 10 cms.
P3 The area of a parallelogram is its base multiplied by height.
C Therefore, its area is 120 sq. cms.

Now, missing premises are not the only source of flaws in derivations. Consider PC6:

PC6: P1 Apollo and Athena are siblings.
P2 Athena and Aphrodite are siblings.
P3 Aphrodite and Hermes are siblings.
C Apollo and Hermes are siblings.

In PC6, we can intuitively arrive at the conclusion from the premises, because we know the meaning of the word *sibling*. We say that in PC6, C follows from P1, P2, and P3. The conclusion is a ***logical consequence*** of these premises.

But now consider these premises:

PC7: P1 Athena loves Apollo.
P2 Apollo loves Zeno.
C **?**

What can we conclude from the two premises? Nothing. Given the premises in PC7, and the meaning of the word *loves*, it does not follow that Athena loves Zeno, nor does it follow that Athena does not love Zeno. Hence, neither of these conclusions would be legitimate.

So, if the conclusion does not logically follow from the premises, we say that the derivation is invalid.

5.3 Circles and Regular Polygons

We are now ready to investigate an interesting question:

Can a circle be a regular polygon?

Mind you, we are not asking whether a regular polygon can ***approximate*** (come close to) a circle. We are asking if there are regular polygons that ***are*** circles. There is a world of difference between the two.

First let us get the terminology clear.

A polygon is ***equilateral*** iff its sides are of the same length.
A polygon is ***equiangular*** iff its angles are equal.
A polygon is ***regular*** iff it is equilateral and equiangular.

(Before going further, visit the Wikipedia entry on regular polygons at https://en.wikipedia.org/wiki/Regular_polygon and read the introductory paragraph, as well as the material under “General Properties.”)

Now let us do a thought experiment.

Draw a regular polygon in your mind.
Inscribe a circle inside that polygon.
Now circumscribe the polygon with a circle.
Gradually increase the number of sides of the polygon.
As you do this, you will see this correlation in your mind’s eye:

Correlation: *As the number of sides increases,*
the distance between the two circles decreases.

To prove that a circle is a regular polygon, you will have to prove that as the number of sides of the polygon increases, the outer circle and the inner circle come closer and closer till they coincide. If they coincide, it means that the circles also coincide with the polygon.

Can you prove (or disprove) the claim that it is possible to do this?

[Clue: To construct that proof, you need definitions of ‘polygon’ and ‘circle’.]

Exercise 5

Think about the axioms and definitions we have discussed so far.

In Euclidean Geometry, there is at least one point between any two points, however small the distance between them. This implies that every line, however short, has infinitely many points.

In the non-Euclidean Geometry of Section 4.2, points can be adjacent, and every finite line has a finite number of points.

TASK: To prove or refute the conjecture that every circle is a regular polygon:
Did you use a Euclidean system, or a non-Euclidian one?
Would your answer change if you use the other system?

5.4 Degrees of Freedom

For a clearer understanding of the two concepts of *circle as a circular line and as a circular disk*, let us try a thought experiment.

Imagine a circular line suspended ten metres above the ground. This line does not form a complete circle; it is only four-fifths of a circle. For a circle to be complete, its two endpoints have to meet.

Now, imagine an ant walking along that line. The ant can go backwards and forwards on the line, but cannot step off the line. If she does — i.e., if she walks from one point on the line towards the centre of the circle — then she would fall off. In this situation, we say that the ant has only one degree of freedom, namely, backwards and forwards. Even if the circular line is a complete circle, such that if the ant keeps walking it will return to where it started from, the degree of freedom would not increase.

Now, imagine that what is suspended ten metres above the ground is not a circular line but a circular disk. The ant can now walk not only backwards and forwards along the boundary of the disk, but also backwards and forwards along any diameter line (in fact, anywhere on the disk), without falling off. Here, we say that the ant has two degrees of freedom.

Let us now consider a spherical surface — say, a spherical balloon. Imagine the ant walking its surface. It cannot pierce through the balloon to get to the centre or to the opposite side. Studying the movements of the ant would be part of the geometry of ***spherical surfaces*** which are two-dimensional.

How about a ball of cooked rice? The ant can burrow through the rice, and get to the centre, or anywhere inside the ball, and even cross to the opposite side. Studying the ant’s movements here would be part of the geometry of a ***ball*** — a three-dimensional object.

[Note: The word *sphere* in ordinary English is ambiguous between a two-dimensional and a three-dimensional object. Mathematicians reserve the term *sphere* for two-dimensional objects, and use the term *ball* for three-dimensional objects.]

You might have already guessed the question that is coming:

> How many degrees of freedom does the ant have
> in the case of the tennis ball and the rice ball? ☺

> To answer this question, we need to understand the concept of 'dimensions'.
>
> Think of a person's age, height, weight, hair colour, and eye colour. These are dimensions (or parameters) along which different individuals may vary. So we can say that in the space created by these parameters, a person has five degrees of freedom.
>
> To do this, we must free ourselves from the need to 'visualise' such spaces created by various dimensions, or to relate them to physical objects from the real world, because they cannot be visualised. They can only be imagined.

5.5 Inventing Geometric Objects

The Geometric objects that we have discussed so far have been:

- Polygons and their subcategories:
  - Triangles and their subcategories
  - Quadrilaterals and their subcategories
  - Regular, Equilateral, and Equiangular Polygons
- Circles and ellipses
- Lines and points

Why did we deal with only these shapes? Why don't we expand the scope of our theory to include the shapes given below, in Fig. 5-5?

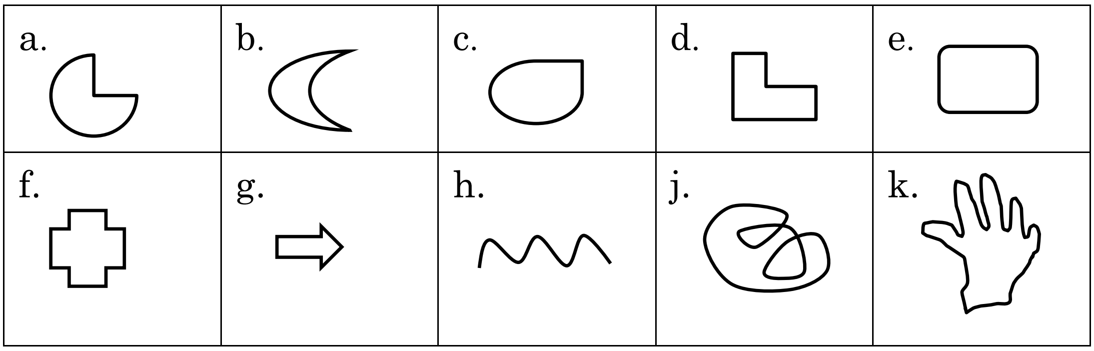


Figure 5-5

There are three possible answers to this question:

Answer 1: Into the worlds of mathematics (such as the world of geometry), we invite only those objects about which we can deduce theorems. For instance, the hand shape in Fig. 5-5k, or the outline of a cow or a bird, do not lend themselves to deducing theorems about them or theorems about their relation to existing geometric objects.

Answer 2: We do have some objects in geometry that we have not talked about. For example, the L-shape (Fig. 5-5d) can be viewed as being composed of a square and two rectangles. And we can then have a formula for its area as the sum of the areas of those three objects. Fig 5-5a can be viewed as being composed of a semicircle and a quarter circle. Its area would then be three fourths the area of the full circle.

Answer 3: It just happens that no one has had the imagination to think up some of these objects as geometric objects.

Now, any form of Rational inquiry calls for reasoning, alongside clarity and rigour of thinking and communication. It is important to be aware that it also calls for creativity — imagination, inventiveness, intuition, and insight. Let us take a look at Answer 3 from that perspective, to explore the role of imagination in the creation of the worlds of geometry.

Suppose there are two curved streets, as in Fig. 5-6.

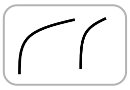


Figure 5-6

And suppose we bring them together to create a joint (a junction/corner/angle) as in Fig. 5-7.

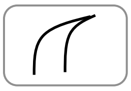


Figure 5-7

We may never have thought of Fig. 5-7 as having an angle. Given that the word *polygon* means 'multiple angles' (*poly* 'more than one', and *gon* 'angle'), we have the option of conceptualising a polygon as a figure with multiple angles, whether or not the lines connecting them are straight. If we make that move, then Fig. 5-5a is a 'triangle' (a closed figure with three angles), Fig. 5-5b is a 'diangle' (with two angles), and Fig. 5-5c is a 'monangle' (with one angle). If so, Fig. 5-5a and 5-5b are polygons, which Fig. 5-5c is not one.

Thinking along such lines would allow us to use our imagination to enlarge the scope of geometry.

5.6 Summing up

Chapter 1 began with the task of constructing a theory of triangles, and Chapter 2 proceeded to a theory of quadrilaterals, beginning with squares, and then going on to rectangles and parallelograms. We then demonstrated the advantages of subsuming triangles and quadrilaterals under the category of polygons.

We then identified the components of polygons as:

POINT:	An intersection between two straight lines
VERTEX:	A joint between two straight lines
SIDE:	All sides of a polygon are straight lines.

Having done this, we constructed a theory of lines and points. That led to a distinction between straight lines and curved lines, which led to a theory of circles and ellipses, which have curved lines, and no vertices.

Further exploration resulted in statements on the relation between polygons and circles. This led to the integration of the theory of polygons and the theory of circles (and ellipses), forming a single theory of geometry.

Such integration is an important function of theories, whether in mathematics, the sciences, or the humanities.

Having gone through Chapters 4 and 5, you would now be able to appreciate the importance of conceptualising the meaning of an academic term in a theory before we try to define it. This means asking ourselves:

What is the concept we need, and are looking for?

Having decided on the concept, we have to ask:

How do we define it?

This distinction between judging the need for a concept, and defining the concept, came up in our conceptualisation of lines and points in Chapter 4, and of circles in Chapter 5.

This process of thinking about concepts and clarifying them is important in all domains of inquiry and research, whether we are talking about parallel lines, velocity, species, democracy, atom, God, or the soul.

For example, suppose we want to assert that 'soul' exists, or that 'soul' does not exist. To do this, it is important that we first figure out clearly what we mean by 'soul'. Asking questions like these can help us clarify our ideas:

Is the concept of 'soul' the same as the concept of 'mind', or is it distinct?
If you think 'soul' and 'mind' are different, what is the difference?

Does the soul exist after the death of the physical body?
If it continues to exist, does it retain the dead person's mental properties?

Does it wander the earth, or go to heaven, or hell?

Does the soul have supernatural powers to influence the physical world?
(For example, can a soul pick up a stone and throw it at a window?)

Without reflecting on such questions and arriving at a decision, it would be meaningless to assert the existence of souls, or to deny it.

The next step would be to define the concept with as much clarity and precision as you can gather, so that you can discuss the issue with others, present arguments in support of or against the existence of the soul, and perhaps engage in a debate on its existence.

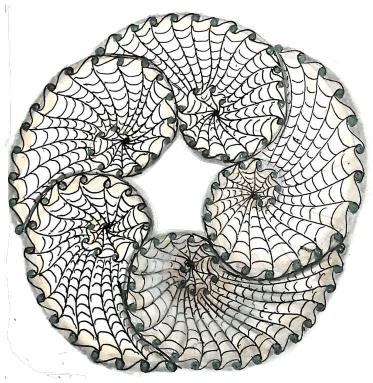
A second learning point in Chapter 5 was the distinction between

the ***properties of an entity*** (e.g., of polygons, of circles)

and

the ***correlation between the properties*** of two (or more) entities (e.g., between circles and polygons).

A third point was the need to revise and refine the statements we learn from books, including textbooks, or from the learning process in school.



In all of this, you have been learning the art and craft of inventing and defining concepts, coming up with conjectures and proofs, and articulating them clearly.

CHAPTER 6:
PULLING THE THREADS TOGETHER

6.1 Constructing and Evaluating Theories: the Art and the Craft

By now, you would have begun to intuitively understand how a theory is constructed. Let us state that understanding explicitly.

The first step when we construct a theory of X is to write down a set of statements that we already know about X. This is what we call a ***description*** of X.

Among the statements, we select some as *premises*. What we derive as logical consequences of the premises, we take as *conclusions* (as mentioned in Section 2.4). We then provide proofs for the conclusions, by demonstrating how they are derived from the premises. This is what we have called the **P**remises-**D**erivation-**C**onclusion (PDC) structure.

Now, we make this move with caution. Our conclusions are tentative, because it may turn out that some of the statements that we initially viewed as conclusions cannot be derived from the premises. We then need to either take them as premises, or add other premises which will help us derive them as conclusions.

Conversely, we may find that some of the statements that we had thought of as premises can be derived from other premises, so we can take them to be conclusions. We saw an example of such a situation in section 4.3, in our discussion of the definitions of parallel lines vs. the theorems that logically follow from them.

The next step is to evaluate each of our proofs. We need to check if each step in our derivations follows logically from the preceding steps and the premises. We ask: are our proofs valid? If there are steps that make our proofs invalid, we need to either *modify our premises*, or *unearth our implicit assumptions* and *formulate them as explicit premises*. This, of course, is likely to increase the number of premises.

As mentioned earlier, for theories outside of mathematics, we need one other component. Scientific theories, for example, require us to discover and establish observational generalisations in the world around us, and show that what we take to be the theoretical premises correctly predict these the observational generalisations. 'Correctly predict' in this context means that the predictions of the theory match the observational generalisations.

Theories of other kinds, including conceptual theories, ethical theories, and theories of value require us to satisfy similar external conditions. Such conditions in those theories would be the equivalents of observational generalisations in scientific theories.

6.2 Abstracting

As you know, geometry is a mathematical theory concerned with shapes in space: the properties of shapes, the relations between shapes, and their relative positions and configurations in space. It is ultimately rooted in our visual experience of the world. When we compare a sheet of paper of A4 size with a compact disk (CD), we figure their shapes to be *rectangular* and *circular* respectively.

Remember, geometry focuses on shapes. Physical objects (e.g., the sheet of paper and the disk) have other properties such as colour, weight, rigidity, opacity, and so on. Geometry ignores these aspects, and extracts the relevant properties of the objects into abstract entities. These ***abstract objects***, which we call *point*, *line*, *circle, rectangle*, *triangle*, and so on, do not have properties like colour, weight, rigidity, and opacity. So the surface of a spherical lemon can be yellow or green, but a sphere has no colour. Geometric objects like spheres are ***abstractions*** that exist in our mind.

The process of abstraction teases apart properties of concrete objects in the world, studies them, and transforms them into abstract properties of abstract objects. Such abstraction is not limited to mathematics. Scientific theories are also built out of abstractions. For example, while we may draw upon our sensory experience of 'force' to construct theories of motion, we subsequently conceptualise and work with an abstract notion of 'force' that might deviate significantly from our experience.

6.3 Generalising and Specialising

Imagine this. We take 10 polygons, and build descriptions of each of them. These descriptions would be the equivalents of what are called 'data points' in scientific theories. Based on the descriptions, we come up with a set of conjectures. We now examine three more polygons, and check if our conjectures hold on them. And then we examine five more polygons. We continue doing this until we can find no ***counterexamples*** (examples that show that we are wrong) to our conjectures. Having satisfied ourselves that there are no counterexamples, we can now proceed to prove the conjectures.

What we have described above is the ***process of generalisation*** from a *sample of polygons* to the *population of polygons*. This methodological strategy of generalising from a sample to a population is useful in any form of collective inquiry, including scientific inquiry.

Remember, in the process of generalising from the sample to the population, we might discover counterexamples. If we do, we may either abandon our original conjecture as false, or see if it can be rescued by limiting the scope of the conjecture to a sub-category of the population.

For example, we might find that all our counterexamples are polygons with concave angles. In this case, we might limit the scope of our conjecture to convex polygons. We might have to restrict the scope even further, for example, to regular polygons. In limiting to particular subcategories, we are using the ***process of specialisation***.

The processes of generalising and specialising are needed in theory building in all domains. For example, we might originally propose a conjecture about vertebrates, but we may find that it applies to animals in general, in which case we *generalise* it. Conversely, we may discover counterexamples, and may need to limit the scope to mammals, in which case we *specialise* it.

6.4 Reasoning, Predicting, Explaining and Proving

As we have seen, theories of geometry are ***axiomatic systems***. This is true of all mathematical theories. In such theories, a conjecture is ***proved*** by showing that it can be derived from the premises (axioms and definitions), using deductive ***reasoning***.

So, mathematical theories involve constructing knowledge through *pure reasoning*, combined with *imagination, insight, and intuition.*

Axiomatic systems have to obey the conditions of ***coherence***. This means that the statements in an axiomatic system must:

a. be logically connected. (***Logical Connectedness***)
b. not contain logical contradictions. (***Prohibition of Logical Contradictions***)
c. cover the widest range of phenomena in its explanation. (***Generality***)
d. contain the fewest possible premises. (***Simplicity***)

Since scientific theories have an axiomatic component, they are also subject to the conditions of coherence. The scientific equivalents of mathematical theorems, as we have seen, are ***predictions*** — the logical consequences deduced from the premises of the theory. But as pointed out earlier, scientific theories have the additional requirement that the predictions must match the observational generalisations. This translates as the requirement that theories must not only make predictions, but those predictions must be correct, where 'correct' means logically consistent with the phenomena that we observe in the real world.

Scientific theories have another function in addition to prediction: that of explanation. The concept of explanation is central to science. In addition to providing answers to the question: "Is this true?", scientific theories also have the function of answering the question: "Why is this true? Why this, but not that?" Answers to such questions provide an explanation, and the understanding that comes from that explanation.

Let us take some examples. Here are a few statements that we take to be 'true statements' (TS). For each of them, we need to ask: "Why is this so?"

TS1: No triangle can have a reflex angle.

TS2: An equilateral triangle is an equiangular triangle, and vice versa.

A reflex angle is more than 180° (half circle) and less than 360° (full circle).

TS3: An angle in a triangle cannot be varied without simultaneously varying the length of at least one of its sides.

To think about

For each of the statements in TS1, TS2, and TS3, figure out if it can be generalised from triangles to polygons, and give reasons for your conclusion.

6.5 Transdisciplinary Thinking

This book has used the domain of geometry to illustrate the concepts and strategies of theory construction, though we have occasionally demonstrated their usefulness in theory construction in biology.

However, as we have said earlier, these concepts and strategies are not limited to geometry and biology, or to any specific domain. They play an important role in the construction and evaluation of theories in all domains of research, ranging from mathematics, and the physical-biological-human sciences, to the humanities. We will therefore refer to them as ***transdisciplinary*** concepts and strategies, using *trans-* in the sense that they cut across specialised disciplines, discipline groups and fields, and exist at a more abstract level.

To take an example of what we mean by transdisciplinary, the ***theory of classical mechanics*** is specific to the highly specialised field that studies gravity and motion within the discipline called physics. However, the concept of ***theory*** is a trans-disciplinary one.

Take another example. The concepts of gravity, velocity, and acceleration are specific to physics, and the concept of skeletal structure, and of cells being composed of biomolecules, is specific to biology. But the concepts of ***compositionality***, ***structure***, and very concept of ***concept*** are transdisciplinary.

In section1.5.3, we made a distinction between ***axioms*** that are specific to a particular field or discipline, and ***general axioms*** that are part of all academic knowledge. *General axioms are transdisciplinary*. We may make the same distinction among definitions. The definition of force as *that which causes a change in the velocity of an inanimate entity* is specific to physics. But the definition of force as *that which causes change* is a transdisciplinary definition, extendable to the animate, even human domains.

To repeat what we said in Section3.3:

Transdisciplinary relations that appear in theories across disciplines include:

Subcategorisation	*x* is A SUBCATEGORY OF *y*
Compositionality	*x* is COMPOSED OF *y, z, …* Variants: *x* is MADE UP OF *y, z, …* *x* is A CONSTITUENT OF *y, z, …* *x* is DECOMPOSABLE INTO *y, z, …*
Ordering	*x* is ORDERED PRIOR TO *y* Variants: *x* is RANKED HIGHER THAN *y* *x* PRECEDES *y*
Logical Consequence	*x* is a LOGICAL CONSEQUENCE OF *y*
Logical Contradiction	*x* LOGICALLY CONTRADICTS *y*
Equality	*x* IS EQUAL TO *y* Variants: *x* IS EQUIVALENT TO *y* *x* IS AN ANALOGUE OF *y* *x* IS A HOMOLOGUE OF *y …*
Correlation	*x* CORRELATES WITH *y*
Causation	*x* CAUSES *y*
Instantiation	*x* is AN INSTANCE OF *y* Variants: *x* is A MEMBER OF set/category *y* *x* is AN EXAMPLE OF *y* *x* is A SAMPLE OF *y …*
Negation	*x* is THE NEGATION OF *y* Variant: *x* is THE OPPOSITE OF *y*

6.6 Admissible Bases for Arguments

The first paragraph of the introductory chapter of the book, *The Works of Archimedes*, by T L Heath begins as follows:

> "A LIFE of Archimedes was written by one Heracleides, but this biography has not survived, and such particulars as are known have to be collected from many various sources. According to Tzetzes he died at the age of 75, and, as he perished in the sack of Syracuse (B.C. 212), it follows that he was probably born about 287 B.C."

Notice the use of "it follows that," in the second sentence. The phrase signals the strategy of ***reasoning*** that forms the core of ***rational inquiry***. We have discussed the methodological strategies of rational inquiry at length in the previous Units. Let us state the premises and the conclusion of the passage explicitly:

Premise 1: Archimedes died at the age of 75.
Premise 1: Archimedes died in 212 BCE.
Conclusion: Archimedes was born in 287 BCE.

Is this conclusion true?

We can answer that question as follows:

> If premises 1 and 2 are true, and the derivation of the conclusion from the premises is valid, then it is true that Archimedes was born in 287 BCE.

This answer tells us that there are two conditions for our accepting the truth of the conclusion:

Condition A: The premises must be true.

Condition B: The derivation of the conclusion from the premises must be valid.

How do we know that the premises are true? The answer is:

> Premises 1 and 2 are asserted by Tzetzes.

But why should we believe that what Tzetzes said is true? An Internet search for the name Tzetzes leads us to John Tzetzes, a Byzantine poet and grammarian who is known to have lived at Constantinople in the 12th century. T L Heath is making the assumption that John Tzetzes' ***testimony*** is a reliable source of information to conclude that Archimedes was born in 287 BCE.

Relying on written testimonies by previous authors has been an important methodological strategy in human history. A similar methodological strategy is used in trials in the criminal court: spoken testimonies of eye witnesses and of expert witnesses are taken as reliable sources of information.

You can see that spoken and written testimonies are not admissible bases for argumentation in the physical and biological sciences. Take the following statement:

> "The gravitational attraction between two bodies is directly proportional to the product of their masses, and indirectly proportional to the square of the distance between them."

The fact that this was proposed by *Isaac Newton* does not make it a legitimate basis to conclude that the statement is true.

Similar remarks apply to the following statement as well:

> All existing and extinct life forms on the earth evolved from unicellular ancestors.

We cannot appeal to the authority of *Charles Darwin* to legitimatise the conclusion that this proposition is true. Expert testimonies are not admissible forms of evidence in mathematics or in the physical and biological sciences.

So, what constitutes an admissible form of evidence in scientific inquiry? The answer is: observational reports. We can treat them as 'eyewitness testimonies'.

6.7 The Nature of Truth in Mathematics

What are the kinds of premises that are accepted as *legitimate bases for argumetation* in mathematics?

In Section 4.2, we saw an example of the contrast between the Euclidean axiom that every finite line, however small, has infinitely many points, and the non-Euclidean axiom that every finite line has a finite number of points, such that the length of a line is the number of points it contains. If we adopt the Euclidean axiom, we deduce the conclusion that every line is bisectable. But if we adopt the alternative axiom, we deduce the conclusion that there exist lines which cannot be bisected.

These theorems may appear to be logically contradictory. However, they are not. Why is that so? This is because mathematical theories are about logically possible imagined worlds. And there can be many such worlds. So what we ought to say is:

> In a world in which the Euclidean axiom of the number of points in a line is true, the theorem that every line is bisectable is true.

But: In a world in which the non-Euclidean axiom of the number of points in a line is true, the theorem that there exist lines that cannot be bisected is true.

There is no logical contradiction between the two because they are about two different worlds.

Similar remarks apply to the geometry of flat surfaces (as in Euclids world) and of spherical surfaces. Euclidean two-dimensional geometry is a geometry of flat surfaces, while Riemannian two-dimensional geometry is a geometry of spherical surfaces. The difference between them is:

IN A FLAT SURFACE GEOMETRY	IN A SPHERICAL SURFACE GEOMETRY
No straight line, regardless of how far it is extended, can meet itself.	Every straight line when extended meets itself.
No two straight lines can intersect at two distinct points.	Any two straight lines when extended intersect at two distinct points.
The sum of angles in a triangle is two right angles.	The sum of angles in a triangle is more than two right angles, and can be upto three right angles.

All this shows that the truth of a mathematical theorem is relative to the theory that it is a part of. Hence, mathematical truths are of the form:

> If such and such premises are true, such and such conclusions are also true.

Are the premises true? Mathematics has nothing to say about that. This is a fundamental difference between mathematical and scientific truths. This is because, unlike theories in mathematics, scientific theories are about the particular world we live in. Hence, the following statements would be logically contradictory:

i) The Earth is the stationary centre of the world; the Sun revolves around it.

ii) The Earth revolves around the Sun, and spins on an axis that is tilted to the plane of its revolution.

These statements are logically contradictory because they are both about the world we live in.

6.8 Summing up

The title of this Chapter, "Pulling the Threads Together," might have conveyed a sense of what the chapter is about: taking the threads — the central ideas — in each of the previous chapters and pulling them together into an integrated whole.

As we said earlier, this book is about the art and craft of constructing and evaluating theories,with its primary focus is on theories of Geometry. But the title of the book is *Constructing Theories: A Case Study in Geometry, with an Excursion into Biology*.

The specification, "with an Excursion into Biology," signals that the book also wanders into the art and craft of constructing and evaluating theories in Biology, although not in such depth or coverage as in Geometry. This is only to illustrate that even though theories in all the different academic domains share a set of common features, each of them may also have features that distinguish them. This is like saying that RATs and ETs share the property of triangleness, but they also are distinct geometric objects.

If the title of this book had been *Constructing Theories: A Case Study in Geometry,* Chapter 6 would have been a summary of the book, and we would not proceed to Chapter 7: "Constructing Theories in Biology," and Chapter 8: "Geometry as Science and Math." So you need to ask yourself at this point:

> What did I learn in Chapters 1-6 through my exploration of constructing theories in geometry, as a special case of constructing theories in mathematics?

We suggest that you think about this question, skim through Chapters 1-6 if necessary, and write down your answer. Do this *before you continue*.

Okay, here is what we hope you have learnt.

Chapter 1 introduced you to the concept of a theory of Geometry as an *axiomatic system*. What is an axiomatic system?

> An axiomatic system is a set of axioms (and definitions) that yield a set of theorems (logical consequences) through reasoning.

In the study of reasoning, called logic, the set of axioms and definitions are called premises, and the theorems are called conclusions. So every axiomatic system has three parts: premises, derivation, and conclusion. In this book, we have used the term *PDC* (Premise-Derivation-Conclusion) *structure* to refer to this aspect of the structure of axiomatic systems.

In Chapter 2, we learnt a different aspect of the structure of theories: that of *categories* and *subcategories*, and how these concepts can be used to put together otherwise separate theories into a single overarching theory.

Take an example. We could have constructed separate theories of RATs and of ETs without bothering about what they have in common, or trying to *connect* them. But we took RATs and ETs as subcategories of the more abstract Triangle, and constructed a theory of triangles instead. This allowed us to derive the special theories of RATs and of ETs by simply adding a few axioms to the existing axioms in the theory of triangles. This also allowed us to avoid duplicating the same statements in multiple sub-theories.

Similarly, in Chapter 3, we could have constructed separate theories of triangles, squares, rectangles, pentagons, and so on. But by treating them as a subcategories of polygons, and constructing an overarching theory of polygons, we minimised the special axioms required for each subcategory. This simplified the larger theory, and explicitly brought out what is shared by all the subcategories.

In Chapter 5, we constructed a theory of circles and integrated it with a theory of polygons, as a theory of geometry (see Section 5.6). Doing this needed a bit of extra work: in Chapter 4, we constructed a theory of lines and points, treating straight lines and curved lines as subcategories of lines.

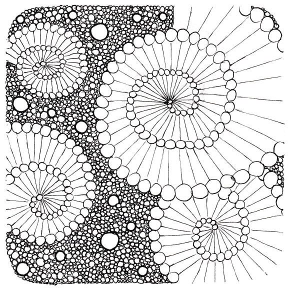
In the process of doing all this, we learnt about the concepts of *conjectures*, *theorems*, and *proofs* in mathematics. A conjecture is a statement that we believe to be true, but have not proved yet. When a conjecture is proved, we call it a theorem.

> Not all conjectures in mathematics have become theorems. In Chapter 3, we read about Goldbach's conjecture. It was proposed by Christian Goldbach in 1742, but has not been proved yet, in spite of the efforts of the world's greatest mathematicians for nearly three centuries. Unproved conjectures are called 'problems' in mathematics. The Wikipedia entry on unsolved problems in mathematics gives a comprehensive (though incomplete) list:
> https://en.wikipedia.org/wiki/List_of_unsolved_problems_in_mathematics

While exploring Geometry as an axiomatic system, we discovered that alternative axioms can yield alternative theories in geometry. Let us take two examples.

(i) Euclidean axioms yield ***Euclidean Geometry***. Euclidean geometry of two dimensional surfaces is a *geometry of flat surfaces*. But if we change the axioms, as in Riemannian geometry, we get what is called ***spherical geometry***. Spherical geometry of two dimensional surfaces is a *geometry of curved surfaces*.

(ii) Euclidean geometry assumes that spaces are gradient/continuous. In these spaces, no two points can be adjacent. Instead, if we assume that space is composed of points such that there exist adjacent points, we can develop a number of ***discrete geometries***.



Thus, geometries of curved spaces and of discrete spaces are both non-Euclidean.

If the idea of alternative geometries intrigues you, you would enjoy reading Ian Stewart's book *Flatterland*.

You may also looking at the Wikipedia entries on:

Taxicab Geometry (https://en.wikipedia.org/wiki/Taxicab_geometry),
Pixel Geometry (https://en.wikipedia.org/wiki/Pixel_geometry), and
Fractal Geometry (https://en.wikipedia.org/wiki/Fractal)

What we have said above does not cover all the important ideas in Chapters 1-6. We leave it to you to take another look at the titles of sections (6.1 to 6.7) in Chapter 6,.skim through them if necessary, and write down what we have not covered in this summary.

We hope you will also start wondering what kind of geometry would treat the four twirls in the drawing above as geometrical shapes to be included in the theory of geometry. Also think of the geometry of non-rigid shapes, such as the shape of a T-shirt, which retains the 'same' shape even when folded and placed in a cupboard, or crumpled in a laundry bag. And the shape drawn on a ballon which retains the same shape when it is squeezed or stretched, or when all the air has been blown out and is left crumpled in a box.

In Chapter 7, we will use some of the ideas of theory construction that we have learnt in theories of geometry to take a peek into the construction and evaluation of theories in biology.

CHAPTER 7:
CONSTRUCTING THEORIES IN BIOLOGY

7.1 Beyond Geometry: Theories in Biology

In Chapters 1–6, we outlined some of the methodological strategies for constructing and evaluating theories, where geometry was the terrain for us to illustrate theory building, and for you to practice it. In this chapter, we will move to a different playground — that of biology — and see how precisely the same methodological strategies can be used there.

In order to show that a conjecture is true, mathematics requires it to be derived from the set of axioms and definitions, thereby establishing them as theorems. Remember that if we move from mathematics to science, we find an additional requirement:

> *In Science, the predictions (the logical consequences) of a theory must be shown to be 'correct'. That is to say, we must ensure that they agree with the observational generalisations relevant to the theory, and furthermore, the theory must explain the asymmetries (why X, but not Y) in them.*
>
> (Section 6.4)

We hope that the move from geometry to biology will highlight the transdisciplinary modes of thinking and reasoning that are ***transferrable*** from one domain of knowledge to another. With that purpose in mind, this chapter will focus on two different theories in biology:

a theory of *anatomy*; and
a theory of *habitat*.

These theories, combined with theories of biological *function*, biological *change*, and biological *development* are essential to the construction of a theory of biological *evolution*. (We won't attempt to construct each of these theories here, as such a task is beyond the scope of this book.)

7.2 Observing, Reasoning, and Formalisms

Suppose you look out of a window and see what is given in this photograph. (Think of the border of the photograph as the frame of the window.)

You see only a part of the entity, not the whole entity, right? Does that prevent you from arriving at certain conclusions?

Figure 7-1

Let us check it out. Will you be able to answer the following questions?

1. Is the entity whose part you are looking at:
   a. inanimate, animate, or neither?
   b. a man, a woman, a child, or none of these?
   c. an animal, a plant, or neither?
   d. a frog, an insect, a tree, a snake, a bird, a bat, or none of these?
   e. a hawk, an eagle, a parrot, a crow, a sunbird, a book, or none of these?

Write down your answers to these questions. And now answer the following 'YES-NO' questions. Again, it would be a good idea to write down your answers.

2. Does the entity in *Fig. 7-1* have the following?
Put a **Y**(es) or **N**(o) in the space to the right of each item.

a.	fur		e.	hair		i.	blood		m.	bones		q.	flowers	
b.	legs		f.	cells		j.	lungs		n.	leaves		r.	feathers	
c.	fins		g.	roots		k.	heart		o.	a beak		s.	nostrils	
d.	eyes		h.	a tail		l.	claws		p.	fingers		t.	branches	

3. Write down your answers to the following questions as well:
   a. If your answer to question (2b) is yes, how many?
   b. If your answer to (2k) is yes, how many?
   c. What else can you say about this entity?

If you now compare your answers with the answers from your friends, a big surprise awaits you: their answers would be no different from yours. And interestingly, all this just because you looked at the picture in *Fig. 7-1*, and recognised it as *a creature with a beak*.

Now, the picture shows only the head. We can observe the details in the picture — for example, (2d), (2o) and (2r). But many other questions cannot be answered on the basis of observation alone. They are ***inferences*** based on what you observe in the picture of the head. These inferences come from the ***knowledge*** *that we already have* (about parrots) in the world around us (e.g., (2b), (2h), and (2l)), on the basis of our ***prior observations*** in the world around us, and from the prior knowledge of birds based on textbooks (e.g., parrots are a subcategory of birds) combined with ***reasoning***.

To arrive at an answer through ***reasoning***, we use a set of statements of the form:	If an entity has X, then it is…… OR it has……
Logicians refer to statements of this form as ***conditionals***:	If X, then Y
In formal logic, logicians use the arrow symbol '→'to express the 'if…then' relation, called ***implication*** (X implies Y).	X → Y

To express ***bidirectional implication*** expressed by 'iff' ('x implies Y' and 'Y implies X'), logicians use a double-headed arrow '↔'	X ↔ Y
In the physical sciences, the corresponding relation is that of ***equality***, expressed by the symbol '=':	X = Y

A note is in order about the equality symbol (=) and the arrow symbol (→) in the formulation of laws. Equations in Chemistry use the arrow symbol to express a **regularity in changes of substances**.

For instance, using the arrow notation:

the equation: $2H_2 + 2O \rightarrow 2H_2O$

says that: Hydrogen (atoms) and Oxygen (atoms) combine to form water (molecules);

and The number of atoms in the expressions to the left of the equality symbol is the same as the number of atoms on the right.

In contrast, laws of motion in physics use the equality symbol to express a **regularity in changes of location**. For instance, the equation in Newton's theory of motion: $f = m.a$

says that

the value of the variables in the expressions to the left and to the right of the equality symbol is the same.
(i.e., if you multiply the number for value of **mass** with the number for the value of **acceleration**, you get the number for the value of **force**.)

In the formulation of laws in the physical sciences (sciences of inanimate entities), the most common *relation* is that *of equality*. For the formulation of laws in the biological sciences (sciences of animate entities), we rely on the *relation of implication*.

7.3 A Theory of Parrots

Let us take a close look at the questions in (1) in the previous section.

In the technical vocabulary of biology, the term ***taxa*** (singular: ***taxon***), denote categories of organisms. The terms *plant*, *animal*, *vertebrate*, *bird*, *fish*, *mammal*, *parrot* and *hawk* refer to taxa of animate entities. The questions in (b-e) are about ***categories*** of organisms: man, woman, child; plant; animal, frog, insect, bird, parrot, hawk, and so on. While the terms 'inanimate' and 'animate' in (1a) refer to properties of entities, the terms 'inanimate entity' and 'animate entity' refer to categories. And the specification, "entities that you are looking at," indicates that the questions in (1) are about categories of entities that we can observe with our naked eyes.

In contrast, the questions in (2) and (3) are about the ***correlations*** between taxa, that is, correlations between categories of animate entities. Based on these correlations, we can infer information about one part of the organism from information about another:

4. a. If the entity has a beak, then ……
b. If an entity is a parrot, then ……

Chances are that your answer to question (1e) is that the picture in *Fig. 7-1* is that of a parrot. Needless to say, having a beak is not sufficient for you to infer that the entity is a parrot. To infer this, you also need some other features that you observed in the photograph.

The set of ***if-then conditionals*** on the basis of which you gave answers to (2) and (3) constitutes a ***theory*** of the anatomical properties of the taxon we call *parrot.* But notice that it is also very similar to a theory of the anatomical properties of the taxon we call *bird.*

7.4 Theories of Birds and Bees

In Chapters 1-6, we illustrated a methodological strategy for theory construction: we construct a theory of X by taking a ***description*** of X and converting it into a Premise-Derivation-Conclusion structure; treating some of the statements in the description as ***premises,*** and others as ***conclusions***; and deriving the conclusions from the premises through ***deductive reasoning***. In Sections 7.2 and 7.3, we extended that strategy to construct a very rudimentary theory of the anatomy of parrots.

This theory begins with the premises in (4), now filled in below, and proceeds to include a number of other premises of the form: If X, then Y:

5. a. If an entity is a parrot, it has a beak.
b. If an entity is a parrot, it has feathers.
c. If an entity is a parrot, it has two legs.
d. If an entity is a parrot, it has two wings.
And so on.

To think about

Do you see that the statement, “If the entity has a beak, it is a parrot,” is false? Can you reason why? ☺

What we have in (5) is not a particularly interesting theory because it postulates a separate premise for each property of the creature. Remember we saw some conditions on axiomatic systems in Chapter 6 (Section 6.4)? One of the conditions was that the premises must be logically connected. Other conditions were that we minimise the number of premises (simplicity), and maximise the range of conclusions (generality).

And remember the idea of the logical inheritance of properties?

> The properties of a category are inherited by its subcategories.
> (Sections 1.4; 3.2))

Given the general principle that *subcategories inherit the properties of their mother categories*, one way of minimising the number of premises is to state the conditionals on higher-level categories. Thus, instead of (5a-d), we may postulate (6a-e):

6. a. 'Parrot' is a subcategory of the category 'bird'.
(i.e., If an entity is a parrot, then it is a bird.)
b. If an entity is a bird, then it has a beak.
c. If an entity is a bird, then it has feathers.
d. If an entity is a bird, then it has two legs.
e. If an entity is a bird, then it has two wings.

We can now deduce all the properties of parrots we have described in (5a-d) from (6a), (6b–e), and the principle of logical inheritance.

This allows for a powerful reduction of premises, because from a single set of axioms on birds, we can now predict the anatomical properties of not only parrots, but also hawks, eagles, vultures, crows, ravens, sunbirds, and so on, when combined with the subcategory statements in (7):

7. a. Parrots are a subcategory of birds.
b. Hawks are a subcategory of birds.
c. Eagles are a subcategory of birds.
d. Vultures are a subcategory of birds.
e. Crows are a subcategory of birds.
f. Ravens are a subcategory of birds.
g. Sunbirds are a subcategory of birds.

Given these subcategory statements, there is no need to duplicate the statement of properties for each of the subcategories separately.

Now, if you compare the properties of birds with the properties of snakes, reptiles, fish, frogs, and mammals, you will find that they have a number of properties in common. Such shared properties across taxa are called ***homologies*** in biology. For example:

8. a. Snakes have vertebrae.
Fish have vertebrae.
Mammals have vertebrae.

b. Snakes have blood.
Fish have blood.
Mammals have blood.

c. Snakes have mouth.
Fish have mouth.
Mammals have mouth.

d. Snakes have eyes.
Fish have eyes.
Mammals have eyes. And so on.

We can achieve economy of premises with the following statements:

9. Vertebrate (DEF): The category of organisms with vertebrae.

10. a. Birds are a subcategory of vertebrates.
b. Snakes are a subcategory of vertebrates.
c. Fish are a subcategory of vertebrates.
d. Reptiles are a subcategory of vertebrates.
e. Frogs are a subcategory of vertebrates.
f. Mammals are a subcategory of vertebrates. And so on.

Now, the different subcategories have not only ***similarities,*** but also ***differences***. That is what the concept of homology implies: similarities among differences, which is the same as homology and diversity, and comes under the broad umbrella of invariance and variability.

We account for both similarities and differences by postulating the similarities on the mother category, and the differences on the daughter category. That leads to the tree structure of biological categorisation:

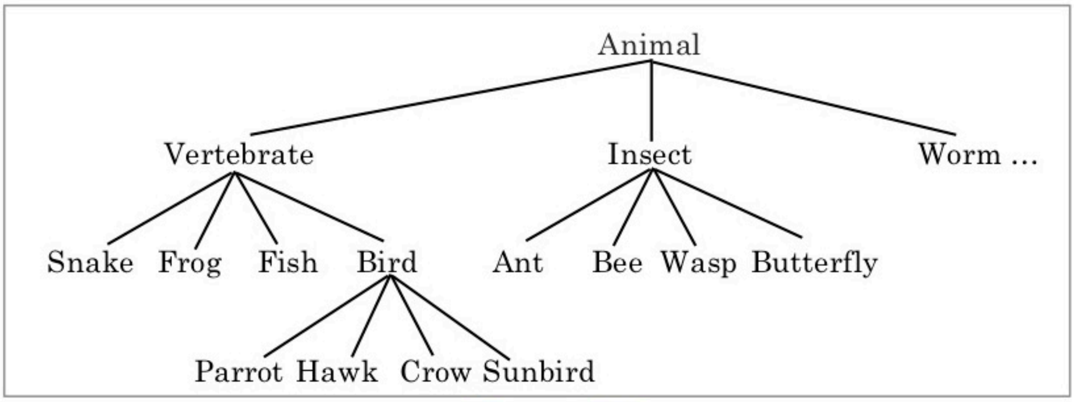


Figure 7-2

To take an example, the Wikipedia entry on butterflies gives us a list of the properties of butterflies. (https://en.wikipedia.org/wiki/Butterfly)

Adult butterflies
- have large, often brightly coloured wings
- have conspicuous, fluttering flight
- have a four-stage life cycle
- undergo complete metamorphosis
- lay eggs on the food plant, on which they grow into caterpillars, which
  - feed on the plant, and
  - metamorphose into butterflies

This is just a short list. The entry does not mention that butterflies

- have compound eyes
- have six legs
- have a neural system
- have a food canal
- have eukaryotic cells
- are multicellular.

The reason why such properties are not mentioned in the entry is because these are predictable from the classificatory system (i.e., a system of categories and subcategories). A theory of the category of butterflies is a sub-theory of the category of insects, which is a sub-theory of the category of animals, which in turn, is a sub-theory of the category of animate entities.

Exercise 1

TASK 1: Read the Wikipedia entries on snakes, frogs, and fish to make a list of their anatomical properties. Derive as many of them as possible by setting up the shared properties as properties of vertebrates.

TASK 2: Read the Wikipedia entries on bees, butterflies, insects, worms, and ants to make a list of their anatomical properties. State the properties of animals to deduce the shared properties of the daughters from those of the mother.

If you now go back to Chapter 6, you will see that the methodological strategies we have used in Sections 7.2–7.4 (and you have practised in *Exercise 1* above) are those of description, abstraction, categorisation, generalisation, reasoning, and prediction.

In the tree diagram in *Fig. 7-2*, we have used yet another strategy, that of ***representations.*** Having seen such visual representations in Venn diagrams (rectangles and circles), and in geometry (diagrams of straight and curved lines, triangles, rectangles, circles, …), it should be easy to connect the different kinds of representations in academic knowledge, and reflect on the IDEAS that the representations express (or are intended to express).

Before going further, it may be a good idea to reflect on this question: Do circles and rectangles in Venn Diagrams represent the same concepts as circles and rectangles in geometry? Your answer is likely to be 'no'. If so, what exactly are the conceptual differences between them? And what are the conceptual similarities? Try to write down your answers.

Exercise 2:

TASK: Read the Wikipedia entries on unicellular life forms, prokaryotes, and eukaryotes, and construct a theory of the anatomical properties of life forms. For this, you will need to incorporate the categories of animate entities, unicellular entities, eukaryotes, and prokaryotes into the tree in *Fig. 7-2.*

7.5 A Theory of the Anatomy of Animate Entities

The theory that we have constructed so far is of the ***anatomical properties of animate entities***. It says very little about anatomical structure as such.

For example, for a ***theory of anatomical structure***, it is not sufficient to say that vertebrates have legs. We also need to say something about the structure of legs. What does this mean? Well, a human leg is composed of the upper leg, the lower leg and the foot. A human arm is composed of the upper arm, the lower arm, and the hand. We can unify legs and arms by stating these structural properties as the properties of limbs. We also need to say something about hands being composed of the palm and fingers, and something similar about feet. Finally, we need to say something about the three-part structure of the digits on the hand, and also, the number of digits.

A theory of anatomy also needs to include internal anatomy (observable only through special instruments or through dissection), for example, the structure of the skeleton, of the heart, of the lungs, and other internal organs; as well as the structure of cells and biomolecules which make up these organs.

Constructing such a theory of the anatomical structure of animate entities can be an exciting and adventurous project. However, it is likely to require sustained work for a few months or more. If you have the time and are game for it, we encourage you to jump into it straightaway.

7.6 A Theory of Functions

The terms hands, legs, eyes, vocal cords, skin, heart, kidney, brain, tissue, cell, cell membrane, nucleus, chromosome, protein, DNA, and so on refer to the parts of organisms. These parts themselves have anatomical structure (part-whole organisation), and participate in systems that perform functions like survival, digestion, metabolism, respiration, circulation, locomotion, reproduction, cognition, and communication.

Theories of structure (anatomy) and theories of function (physiology) are subtheories of a theory of biology. One of its components is the theory of the relation between the two subtheories. Take, for instance, the function of reproduction found in every taxon, ranging from bacteria to humans. But the way this function is manifested in different taxa exhibits considerable variation. Unicellular organisms as well as the cells in multicellular organisms, for instance, reproduce by splitting, or cell division. Viruses reproduce by infecting the bodies of organisms with cells, and letting the reproductive mechanisms of cell-based organisms reproduce them.

How do plants reproduce? How do insects reproduce? How do birds reproduce? How do mammals reproduce? Contructing a theory of the diverse types of reproduction is a fascinating project in Biology.

We would like to suggest a project that has the potential to develop into a research project, beginning with:

a. reading of the following articles:

~ "Plant and Animal Reproduction." In the National Geographic Magazine Education (https://education.nationalgeographic.org/resource/plant-and-animal-reproduction/8th-grade/)

~ "Reproduction." In The Encyclopedia Britannica (https://www.britannica.com/science/reproduction-biology).

b. drawing a tree diagram for the evolution of reproduction in animate entities.

Another valuable inquiry-research project that can help in developing the capacity for theory construction involves exploring the evolution of cognition in animate entities. All life forms from bacteria to humans have some form of cognitive capacity to construct knowledge of the world they live in, in order to survive and flourish. In humans and other vertebrates, the central nervous system is the primary anatomical substrate for this function, even though it is now well known that gut microbiomes also shape cognition in humans and perhaps other life forms as well. See "Microorganisms in the gut are linked to cognitive function."
(https://www.medicalnewstoday.com/articles/microorganisms-in-the-gut-are-linked-to-cognitive-function)

Human cognition requires not only the mechanisms of the brain to process information from the world, but also the mechanisms of the sense organs, including the eyes (light, vision), the ears (sound, hearing), the nose (smell), the tongue (taste), the skin (touch: temperature, texture), and the muscles attached to the skeleton (weight, degree of resistance…) In bacteria, sense organs are molecular, called organelles. Further in the project, do an Internet search for topics like bacterial cognition, plant cognition, insect cognition, bat cognition, cognition in dogs, octopus cognition, and human cognition. Then draw a tree diagram for the evolution of cognition.

7.7 A Theory of Habitat

In the fifth edition of the book *Origin of Species*, published in 1869, Charles Darwin uses the term 'Survival of the Fittest', to mean "better designed for an immediate, local environment." We can interpret this idea of 'fitness' in two ways:

11. A. Only those species that are *fitter than all other species* are selected for survival. All other species become extinct.

B. There is a *threshold* below which a species is considered unfit. Species that are unfit become extinct. All other species survive, regardless of their *relative fitness.*

If we need to choose between A and B, we have to critically evaluate and compare the predictions of (A) and (B). And to do that, we will have to construct a theory of fitness.

Such a theory is beyond the scope of this book. So without going into details, let us say that:

12. For organisms (a single organism/variety/species/taxon) to be fit, their anatomy, physiology, and behaviour must be adapted to their habitat.

If they are not adapted to their habitat, they are unfit, and will become extinct.

The next step is to specify the parameters relevant for the fit between the anatomy, physiology, and behaviour of organisms on the one hand, and their habitat on the other. And for this, let us define habitat as follows:

13. ***Habitat*** (DEF): The habitat of organisms is the local three-dimensional space in which they exist.

The space where organisms exist includes the properties of the space surrounding them, and whatever else exists in that space (for example, other organisms).

Let us take a few examples.

14. Imagine the habitats below; each of them is a space enclosed in a glass case:

	Filled with	SURVIVAL / EXTINCTION		
		Fish	Mouse	Earthworm
A	water, but no air or soil	survive	die	die
B	air, but no water and soil	die	survive	die
C	moist soil, but no water or air	die	die	survive
D	moist soil below and air above the surface	die	die	survive

In order to explain such *experimentally testable correlations* between

a) organisms and their habitats, and

b) their extinction,

we need to understand

i) their anatomy, physiology, and behavior; and

ii) their habitat.

For example, what are the anatomical and physiological properties that make fish survive in habitat A but not in B-D? Another question one can explore is, what are the anatomical and physiological properties that make fish survive in habitat A, but not mice or earthworms? What are the cellular and molecular bases of these anatomical and physiological traits? Such questions would need to be answered in order to arrive at a coherent theory of fitness that makes testable predictions on why certain organisms become extinct in some habitats, and why others become extinct globally.

The parameters of water, air, the earth's surface, and under the earth as broad categories of habitat are not sufficient to explain all the correlations between organisms and their habitats. For each one, we also need to specify:

15.a. Temperature: The upper and lower boundaries for the range of temperature in which the organisms can survive (For example, bacteria can survive in temperature ranges in which mice would not survive).

b. Composition: The molecular composition of the substances (Oxygen, Nitrogen, Carbon Dioxide in the air, water, and soil).

c. Nutrients: Availability of nutrients (plants for cows; animals for lions)

d. Symbiotic Relations: Relation of mutual dependence between species. (For example, bees depend on flowering plants for nutrients, and these plants depend on bees for pollination.

Exercise 3

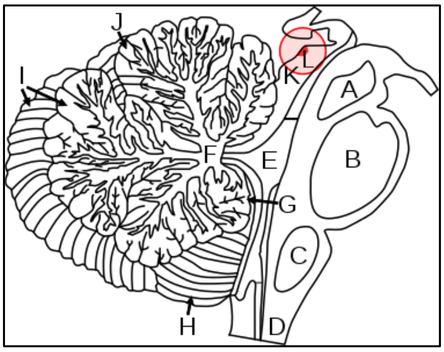
TASK: Choose one category of living creatures (e.g., ant, butterfly, crow, mouse, insect, bat, tiger, bird, and so on). Read the Wikipedia entry on its habitat, and describe how well-adapted it is to its habitat in its anatomy, physiology, and behaviour. What are the conditions this category requires for it to survive, and under what conditions do you expect it to become extinct?

Exercise 4

The chapters in this book began with theoretical geometry and then proceeded to theoretical biology. Now, both geometry and biology use the language of diagrams. For instance, take the discussion of cognition in section 7.6, and the centrality of the structure of the brain in human cognition. Any chapter on the human brain even in a school textbook has diagrams, which makes you wonder about the geometry of cognition, as well as the geometry of biology.

Take for instance, the diagram of the brain in the entry for what is called "Superior Colliculus," reproduced here for convenient reference.

SUPERIOR COLLICULUS



https://en.wikipedia.org/wiki/Superior_colliculus

Does this diagram make you wonder about the various drawings that pop up randomly all through this book, without any obvious function? Have you wondered about their relation to the other drawings and diagrams in the book: of triangles, rectangles, and circles; tree diagrams; and the diagram above?

Have you wondered about a possible theory of the geometry of biology, and ultimately, a theory of the geometry of the physical, biological, and human worlds?

If the questions in Exercise 4 arouse your curiosity, keep thinking about them throughout your school and college education. And continue thinking about them even afterwards. And if tey bring you delight, share it with others, especially young learners.

7.8 Summing up

In an article in the journal *Nature*, Nobel Laureate Paul Nurse points out that biology needs to go beyond mere data and descriptions to ***ideas*** that can explain and predict what we find in the data and descriptions.

Paul Nurse, "Biology must Generate Ideas as well as Data."
Downloadable at: https://www.nature.com/articles/d41586-021-02480-z

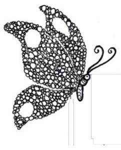
What we have done in this chapter may be viewed as a way of following Nurse's recommendation, by demonstrating what it takes to generate ideas and develop them as theories whose predictions are testable.

With that remark, we will leave the rest of the "Summing-Up" to you. What we would like you to do is to go through all the chapters so far, and write a summary. When you do that, make sure to specify what you learnt from it that is of value to you, and will be of value in your future work. You might find that organising your summary in terms of the following questions would help you to structure your thoughts:

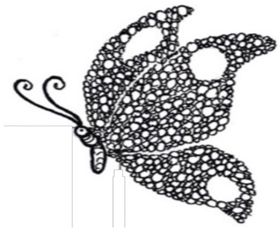
a. In the chapters so far, what did you learn about theories and theory construction (a) in geometry; and (b) in general?

b. In the chapters, what did you find exciting? Was there anything particular that triggered in you a deeper interest in it?

c. Is there anything in these chapters that you did not understand? How can we help?



We have a request. Please send us that summary at tara.mohanan@gmail.com. We would like to find out whether or not our efforts have benefited you, and how we can revise the book to benefit more learners, and provide better value.



Chapter 8:

Geometry as Science vs. Geometry as Mathematics

8.1 Preliminary Remarks

Let us now move towards gaining an experiential understanding of the distinction between *mathematical* and *scientific* theories. To this end, we will explore two-dimensional Euclidean Geometry, first as a science grounded in what we observe in the world around us, and then as an axiomatic system in mathematics.

That exploration would lead to a somewhat unconventional conceptualisation of two-dimensional Euclidean geometry as a scientific theory that explains and predicts a specific set of properties of the shapes of objects. When we see a rectangular tabletop, and a circular plate on it, we conceptualise their shapes as abstract objects called *rectangle* and *circle*. We conceptualise corners as *vertices*, pieces of string as *lines*, dots on a paper as *points*, and so on. We describe these abstract shapes in terms of:

abstract properties of geometric objects	such as *length*, *straightness*, and *angle*;
the relation between those properties	such as between: *vertices* and *angles* of *triangles*; or the *circumference* and *diameter of circles*;
the relation between two or more objects	such as between *triangles* and *circles*.

As an example, let us take lines. Lines have properties that can be described in terms of length, and straightness. In Euclidean geometry, a straight line cannot intersect itself. If the line is not straight, it can be either intersecting or non-intersecting. It can also be either open or closed. Similar remaks apply to objects like triangle, quadrilateral, and pentagon — whether or not they are equiangular, and whether or not they are equilateral, and so on.

Within this perspective, we may say that ***the axiomatic component of any theory*** is the system of axioms and definitions, conjectures and theorems, and the derivation of theorems from axioms and definitions. This is true of the axiomatic component of a scientific theory as well.

In the next section, we will tell a story that may help us in articulating clearly our understanding of the relation between theory construction in mathematics and in science. What we have to say is at the very heart of *ways of thinking and understanding* in these two kinds of academic adventures.

8.2 A Tale of Two Communities of Investigators

Take a cotton thread, dip it in ink, and throw it on a piece of paper. You will probably get a squiggle of the kind in *Fig. 8-1*.

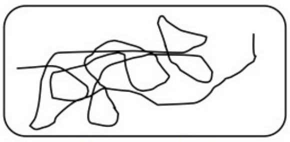


Figure 8-1

Next, take *two* threads, dip each one in ink of a different colour, and throw them both on a piece of paper, one of them on top of the other. You will see the colours cross each other at several locations.

Now take a cotton thread, dip it in ink. But this time, hold the thread at the two ends, stretch it tightly, and hold it against the paper. The mark will be as in *Fig. 8-2*.

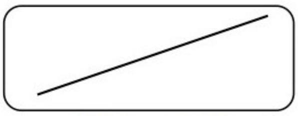


Figure 8-2

Again, take *two* threads, and dip each one in ink of a different colour. If you make marks with them on paper, with the two threads stretched tight, you will never find them crossing at more than one place.

Why should stretching the threads tightly have this effect on their crossing each other? Does this remind you of something you have learnt before?

Look out for other interesting patterns that you might observe in the behaviour of threads. Here is one.

Suppose you take a thread (no ink), and attach one side of it to a drawing pin fixed on a board. Attach the other side to a pencil, and with the thread stretched taut, let the pencil make a mark going round all the way until it comes back to where it started. You will get *Fig. 8-3*. Call this thread the 'Radius thread.'

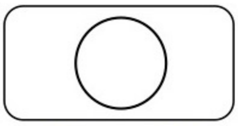


Figure 8-3

Now suppose you measure the length of the Radius thread. Call it length R1. Place another thread on top of the circular mark in *Fig. 8.3*, matching it exactly, without stretching it, till the two ends meet. Measure that length. Call it C1. Now find the ratio C1/R1.

If you do this for several threads of different lengths R and C, you will find that the ratios C1/R1, C2/R2, C3/R3... are the same. Why should this ratio be constant? Can you figure it out?

> If the above instruction to draw a cirle with a thread is not clear, watch these two videos:
>
> "Life Hacks Ways To Draw a Circle Without Compass," at https://www.youtube.com/watch?v=_yqmqESN_Oo
>
> "How to Draw an Ellipse Using the Pin and String Method," at https://www.youtube.com/watch?v=JvciWGTdBq8

Let us now go to the story we promised to tell you.

Once upon a time, long before Euclid, a group of individuals were interested in investigating the properties of threads. They called their study 'threadology'.

Threadology focused on the *length* of threads and their *behaviour* depending on whether or not they were stretched taut, and the *relation* between such threads. They were not interested in properties like thickness, colour, or breakability.

As part of their investigation, they studied the behaviour of threads in situations described in Figs. 8-1, 8-2, and 8-3. They found that when two threads were stretched taut, they didn't cross each other more than once. They also found that for different lengths of threads, the ratio C1/R1, C2/R2, C3/R3... was the same.

In the very next village was another group of investigators who studied the properties of ropes. They called their study 'ropology'. They found that when two ropes were stretched taut, they didn't cross each other more than once, exactly as in the case of threads. Going through the same kinds of processes as the threadologists, the ropologists found that for different lengths of ropes, the ratio C1/R1, C2/R2, C3/R3... remained the same, exactly as in the case of threads.

One day, the threadologists and the ropologists happened to meet. When they talked about their investigation, they were astonished! The general patterns they observed with threads and ropes were the same. Why should this be so?

The two groups happily merged into a single community. They studied the properties of both threads and ropes. Combining threadology and ropology, they called their study 'thropology'.

To figure out what was common to threads and ropes, they ignored the *differences* between them, and abstracted out *what they shared*. This, they called a 'line'. They described their preoccupation as working out a coherent story of lines such that they could correctly predict and explain the properties of lines and their interactions.

Conceptualised this way, they saw thropology as a scientific discipline that sought to predict and explain certain aspects of their *observations* on threads and ropes. They did not see this as a branch of mathematics.

Is it possible to connect their preoccupation to Geometry, and to mathematics? We will engage with that question in Section 8.5.

8.3 Expanding the Scope of Thropology

Our story continues. Unknown to the thropologists, there was another group of investigators whose interest lay in observing how paper can be folded, and studying the creases created by folding. They called their study 'creasology'. Let us try to reconstruct what they did with sheets of paper.

Take a sheet of paper and fold it randomly, just once, running your thumb nail along the *fold* so that it is crisp. Now unfold it. You will see a *crease* on the paper, like the one in *Fig. 8-4*, right? Now make another fold, and unfold it, such that the paper has another sharp crease.

Figure 8-4

Can you do this in such a way that the two creases cross each other?

Can the creases cross each other at two distinct locations? It won't take you long to figure out that the answer is, 'No.' Do you see that this pattern is exactly what the thropologists found in the behaviour of threads and ropes stretched tight?

The creasologists went through all the same experiments, and found another pattern. They took a sheet of paper P1 and folded it. Then, without unfolding it, they folded it again, such that one part of the first fold lined up exactly with the other part. They took another piece of paper P2, and repeated this kind of folding. When they placed the two folded pieces of paper one on top of the other, they found that they were able to place them in such a way that the two folds of P1 and of P2 aligned exactly with each other. No matter how many times they did this, no matter what the colour or size of the paper, no matter what the orientation of the folds, they were aligned perfectly.

Figure 8-5

The creasologists described all their findings to anyone who would listen. When the thropologists heard about this, they were intrigued. That no two creases can cross each other at two distinct places was similar to what they found in threads and ropes. They met with the creasologists, and discussed each others' findings.

They found many patterns that threads and ropes have in common with creases on paper. But there were also important differences. The creasologists had no way to create the image in Fig. 8-3, because they could not create curved creases by folding paper. And the threadologists did not have a procedure to get the configuration of perpendicular creases that the creasologists got by folding the paper twice as described above (Fig. 8-5).

There was another important difference. Creasologists had a way of ensuring straighness of edges, but unlike thropologists, they did not have a way to measure length.

Among the thropologists, some were interested in land measurement. For this, they used ropes to measure the distance from one point to another. That was the shortest path between the points, and the shortest path was a straight line. Their primary interest was finding the least distance, not the straightness of a line.

Other thropologists were interested in ensuring straightness of edges, in carpentry, and architecture. They needed to create windows with window frames, for instance, and the edges of both needed to be perfectly straight, and aligned. A line drawn without being guided by a rope or thread tightly stretched may not turn out straight. Once the straight line was drawn, they had ways of measuring the length.

In spite of these differences in their starting points, they discovered that the abstractions underlying ropes, threads, and creases were point locations, straight lines and circles.

So the thropologists and creasologists agreed that the behaviour of threads and ropes when stretched taut and of creases made on paper by folding are examples of the same abstract pattern. They also understood that the two communities were pursuing the same set of questions, although with different objects: threads, ropes, and paper. So, they set out to unify the knowledge they arrive at, as they realised that such unification could lead all the groups to further understanding.

They decided to use the term ***straight line*** to refer to the abstract concept shared by stretched threads, stretched ropes, and folds and creases on paper; and ***non-straight line*** for the threads and ropes that are not stretched. They now had two kinds of lines, straight and non-straight.

They then discovered that if they placed the tip of a pencil on a piece of paper and dragged it, the result would be a line, either straight or non-straight. When they dragged the pencil tip along the edge of a folded paper, the result was always a straight line. But when they dragged it along the edge of their fingers placed on paper, they got a non-straight line. And when they dragged it along the edge of a disk, the result had the same properties as in *Fig. 8-3*.

By now, there was a large community of investigators exploring the properties of lines using threads, ropes, paper folds and creases, and the edges of objects. Those who had been working on folding paper discovered that they could create geometric objects with three straight lines, which they called ***trilaterals***; with four straight lines, which they called ***quadrilaterals***; with five straight lines, which they called ***pentalaterals***; and so on.

By now, their inquiry had become quite sophisticated. They began classifying the entities they were investigating along different dimensions, as in *Fig. 8-6* and *Fig. 8-7*:

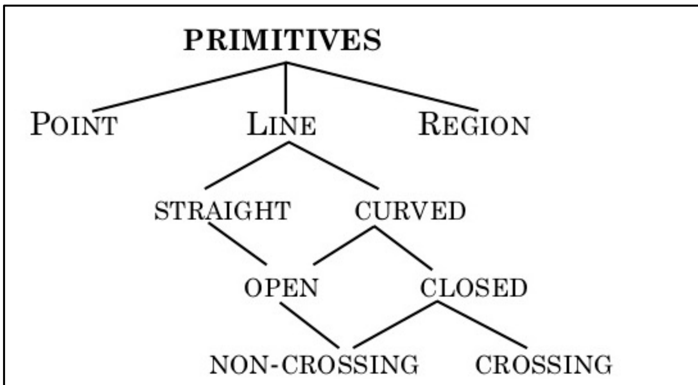


Figure 8-6

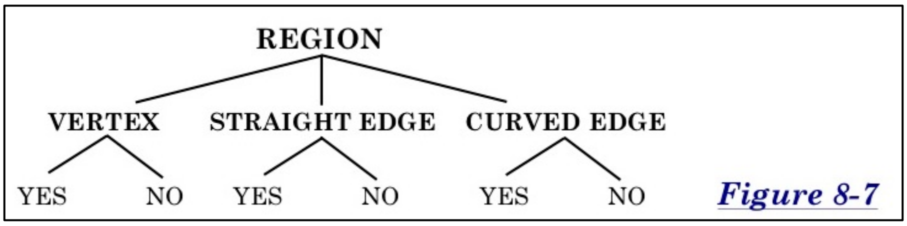


Figure 8-7

Notice that the tree representation in *Fig. 8-7* does not allow us to capture certain combinations, while the 'feature' representation in *Fig. 8-8* below facilitates cross-classification, and also reveals a gap:

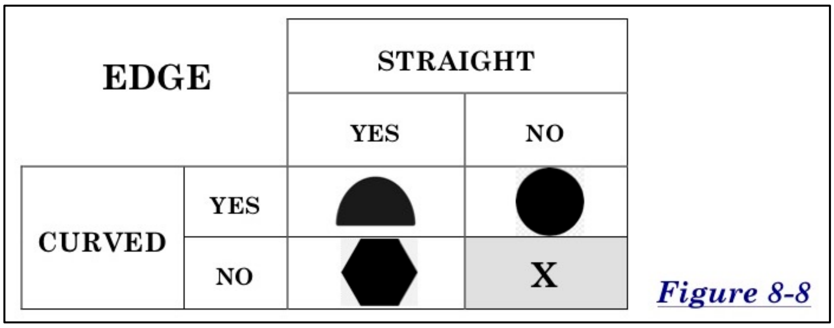


Figure 8-8

The investigators now came up with the idea of what they called ***angles***. "Take a straight line AB," they said. "Rotate AB around A such that B moves to B'." The degree of rotation from AB to AB', they called an angle. And the joint between lines AB and AB' forming an angle, was a ***vertex***.

They conceptualised further: When they rotated AB all the way such that B returned to its original location, they called it a ***full rotation***. If the rotation was one-fourth of a full rotation, they called it a ***right angle***. If it was half of a full rotation, they called it a ***straight angle***: *straight* because when BA and AB' are joined together, they formed a straight line BAB'.

They then found that the number of sides in a closed figure was the same as the number of angles in it. Using the suffix *–gon* ('corner'), they classified the shapes into ***trigon*** (what we now call *triangle*), ***quadrigon*** (*quadrilateral*), *pentagon*, *hexagon*, and so on. If a polylateral (many-sided figure) had N sides, it had N angles, and vice versa. So, the terms *polygon* and *polylateral* were names for the same geometric object.

Connecting the terminology in the story to what we are familiar with today:

X-angles are called X-*gons* in Latin. Hence the names ***polygon***, ***pentagon***, ***hexagon***, and so on. But instead of 'trigon', we use the label ***triangle***, and instead of 'quadrigon', we say ***quadrilateral***. (By the way, in Sanskrit, the word for *triangle* is *trikoNa*; *tri-* and the English *three* come from the same root, and so do *–kona* and the English *corner*.) The mixed-up terminology of English is something we will have to live with. ☺

To continue with some familiar terms, polygons in which all the angles are equal are called ***equiangular*** polygons, and those in which all the lines (called 'sides') are equal are called ***equilateral*** polygons. And polygons that are both equilateral and equiangular are called ***regular*** polygons.

By this time in the story, the investigators had a large number of what they called ***observational generalisations.*** These were statements that they considered to be true of all the members of a category (a population). Some statements were about the relation among members of two or more categories.

For example, the 'angle-sum hypothesis' was an observational generalisation: it said that the sum of angles of a triangle was a straight angle. Another observational generalisation said that an equiangular triangle was also equilateral, and vice versa.

Yet another observational generalisation said that for every triangle, there existed a circle that passed through every vertex of the triangle; and there existed a circle that, if placed inside the triangle, touched every side of the triangle without crossing it.

The most famous observational generalisation, the Pythagoras Generalisation, resulted in the Pythagoras Theorem, a story we will reserve for another time.

8.4 Knowledge, Knowledge Claims, and Proofs

At this point, the investigators became sensitive to the distinction between *knowledge* and *knowledge claims*. A ***knowledge claim*** was a statement that they believed to be true, but had not yet been established as part of knowledge. To be established as ***knowledge***, they said, the claim must be proved.

They turned their attention to the challenging task of proving the knowledge claims they had arrived at. The first such claim was the following statement:

> No two straight lines can meet or intersect (cross) at two distinct points.

Creasologists working on paper folding set up a research project in which they asked many researchers to fold a piece of paper twice such that the resulting creases cross at two distinct points. Even after doing this with a sample of more than 10,000 pieces of paper, they did not find a single pair of creases that crossed at two distinct points.

They offered the following proof:

Conjecture: No two straight lines can meet or cross at two distinct points.

Reasoning: *Sample-to-Population*

Proof: We have observed a large sample (more than 10,000) of pieces of paper with two creases on each piece.

We have not found a single piece of paper in which the two creases meet or cross at two distinct points.

Therefore, until we find evidence to the contrary (that is, a pair of creases that meet or cross at more than one point), we conclude that no two creases can meet or cross at two distinct points.

Can this be taken as a proof for the claim that in the entire population of creases on paper, no two creases can meet or cross at two distinct points?

No. The most crucial phrase in the conclusion is:

"…until we find evidence to the contrary."

It acknowledges the sense of fallibility of the conclusion, and the willingness to correct it in the face of counterevidence. This is an important characteristic of scientific inquiry.

Unlike scientific proofs, mathematical proofs call for conclusiveness, meaning absolute certainty. This requires demonstrating that situations that logically contradict the conclusion do not exist.

Scientific proofs like the one above use the form of reasoning called 'defeasible reasoning'. This is a form of reasoning that mathematical proofs do not allow.

To turn to another issue, notice that the proof above has to do with creases. But our conjecture was not about creases, but about straight lines. Remember the investigator groups had decided to use the term ***straight line*** to refer to the **abstract** concept shared by stretched threads, stretched ropes, and folds and creases on paper? Since the conjecture is stated on 'straight lines', the demonstration that no two creases can meet or cross at two distinct points is not a proof of the given conjecture: it does not extend to stretched threads, stretched ropes, or tips of pens dragged across a straight edge on paper, all of which fall under the abstract concept, 'straight line'.

Those who worked on threads, ropes, and pen marks on paper also examined large samples. None of them found a single instance where two threads, ropes, and pen marks on paper crossed or met at two distinct points. How would they come up with a single proof that applied to these different types of straight lines?

To come up with a proof of that kind, they had to first postulate an abstract entity called ***line***, and then define the concept of 'straightness of line'. The researchers came with some ideas:

Postulates (POSs):

POS-1: A **LINE** is an abstract entity that has length, but no breadth or thickness.

POS-2: An **INTERSECTION** or **JOINT** between two lines is a **POINT**. If a line has finite length, its two ends are also points.

POS-3: A line between two points is straight if and only if it is the shortest path between them.

Using these postulates, they were able to come up with a proof, which if valid, was conclusive; that is, it had total certainty, and ruled out the possibility of, say, a pair of two straight lines that could be a million kilometers long and met at two distinct points.

Exercise 1

Using postulates 1-3, come up with a conclusive proof of the conjecture:

No two straight lines can meet or cross at two distinct points.

Now, there was another conjecture that they were intrigued by. The rope researchers came up with a claim that the sum of angles in a triangle is equal to a straight angle. For this, they decided to measure the angles. For this, they used the concept of ***degree***. They divided a full rotation into 360 equal parts, with each part being a degree, so that a full rotation is 360 degrees. They invented an instrument called a 'protractor' that they calibrated in such a way that a straight angle was 180 degrees, half of a full rotation. This instrument allowed them to count the number of degree marks on it to measure angles.

To construct a rope triangle, they needed six people: one to hold each side of a rope tightly stretched, with three such ropes. And a seventh person measured the three angles with a protractor. They measured the angles of a large number of such triangles, and added together the degrees of the three angles of each triangle. When they did this, and calculated the mean by dividing the total number by the number of triangles, they found the mean to be around 180 degrees.

Using this idea, they offered the following proof:

Conjecture: The average sum of angles in a triangle is 180 degrees.

Reasoning: *Sample to Population*

Proof: We have observed a large sample (more than 10,000) of rope triangles and measured their angles.

The mean of the sum of the angles in a triangle is 180 degrees, plus or minus two degrees.

We have not found a single triangle in which the sum is less than 178 degrees or more than 182 degrees.

Therefore, in the absence of evidence to the contrary, we conclude that the sum of angles in a triangle is not less than 178 degrees, and not more than 182 degrees.

The creasologists also came up with similar proofs for corresponding knowledge claim in creasology.

We saw that the proof given earlier for the claim that no two straight lines meet or intersect at two distinct points was not a mathematical proof. Likewise, the proof given above for the claim that the sum of angles in a triangle is 180 degrees was not a mathematical proof either: it did not rule out the possibility that there existed a triangle somewhere whose sides are a thousand kilometers long, in which the sum of angles is more than 182 degrees or less than 178 degrees.

8.5 Looking Back

This chapter tells a story about ancient researchers who were trying to solve the practical problems of land measurement, carpentry, and the construction of buildings. Different groups first developed theories called threadology, ropology, and creasology, to discover that, at an abstract level, the problems they were trying to solve were the same. Hence they decided to integrate their theories into a single theory of geometry that postulated the concepts of points, lines, and other geometric objects.

In their journey, they received experiential knowledge from more experienced individuals in each group. This was knowledge that they had come to rely on in their practical life on the basis of the memory of their experience. But their ongoing work involved going beyond that knowledge, to a form of knowledge that was academic, where they were expected to 'prove' or 'rationally justify' their knowledge claims.

The kind of reasoning that these researchers used in their justification of knowledge claims can be described as what is called ***Inductive Reasoning***, which seeks to jusify a knowledge claim about a population on the basis of a sample from that population. This form of reasoning is characteristic of the justification of observational generalisations in scientific inquiry. We also pointed out that this form of reasoning lacks the kind of certainty required in mathematics.

Having used inductive reasoning from observational reports to arrive at conclusions when engaging with geometry as a science, if we wish to move to geometry as mathematics, we need to also shift to deductive reasoning from axioms and definitions. That shift will be the subject of discussion in the coming chapter.

Chapter 9:
Figures, Shapes, Maps, and Geometries

9.1 Geometry as an Axiomatic System

Let us continue with the story of the thropologists and creasologists in Chapter 8. Those researchers made ***observational reports*** on a large random ***sample*** that was ***representative*** of the ***population*** that they were exploring, and used those reports as premises to establish the ***observational generalisations***, which in turn would serve as the premises for building a theory of the phonomenon.

In short, the proofs in that story use the form of reasoning called 'Sample-to-Population Reasoning', which uses what is called *inductive logic*, to establish ***observational generalisations***. The premises used for this are ***observational reports*** on a sample. This kind of proof is called an ***Empirical Proof*** — the kind of proof that scientific inquiry employs.

In the land of creasologists and thropologists, there was a young scholar called Daza who had been studying various logics, including ***deductive logic***. She suggested a different way of proving knowledge claims. She called her proofs ***Axiomatic Proofs***. Daza did not know about it in her times, but we know now that mathematical inquiry employs axiomatic proofs.

Mathematics is an ***Axiomatic System***. Axiomatic systems are configurations of *axioms, definitions, their logical consequences,* and *the derivation of those consequences*; and they use axiomatic proofs.

An axiomatic proof is the demonstration that what we think is a conclusion, arrived at from the axioms and definitions, is indeed a logical consequence (called a ***theorem***) of the axioms and definitions. So while the ultimate premises of an ***empirical system*** are ***observational reports***, those of an ***axiomatic system*** are ***axioms*** and ***definitions***.

Reasoning in Axiomatic and Empirical Systems

The only form of logic that an **axiomatic system** uses in order to establish truth claims is **classical deductive logic.**

An **empirical system** also uses classical deductive logic. It also uses other systems of logic, including **inductive logic,** which captures ***sample-to-population reasoning***.

Daza wanted to define the concepts found in thropology and creasology. She began by 'defining' the entity she wanted to call 'line'. What she came up with was a description. It did not qualify as a definition. But using this concept that she took as a primitive (undefined), Daza proceeded to define the other concepts she needed:

1.	*Line:*	an object that exists in a space. It has length, but no breadth or thickness.
2.	*Point:*	an object that exists at the ends of a line, with no length, breadth, or thickness. It is shared by two lines that meet or cross.
3.	*Straight line:*	the shortest path between two distinct points.
4.	*Polygon:*	an object made up of only straight lines forming a closed figure.
5.	*Triangle:*	a three-sided polygon.
6.	*Quadrilateral:*	a four-sided polygon.
7.	*Parallelogram:*	a quadrilateral in which the opposite sides are parallel.
8.	*Rectangle:*	an equiangular parallelogram
9.	*Square:*	an equilateral rectangle
10.	*Circle:*	a closed line in which every point is equidistant from a central point.

She also proposed a few axioms, some of which were specific to geometry (A-C), and others that were more general (D-E):

AXIOMS

A. *Unique Path:*	For any two distinct points A and B, there exists one and only one shortest path between them.
B. *Non-collinearity:*	In a polygon, no three vertices can be colinear.(i.e.: existing in a single straight line)
C. *Openness of straight lines:*	No straight line, however extended on both sides, can meet itself.
D. *Acceptance of Logical Consequences:*	If we accept a set of premise propositions as true, we must also accept their logical consequences as true.
E. *Prohibition of Logical Contradictions:*	Logically contradictory propositions are disallowed in a body of knowledge.

Daza knew that some of the definitions contained concepts which themselves needed to be defined. For example, in the definition of 'point', what is the concept of length, also implicit in the reference to the *shortest* path in the definition of 'straight line'? She was also aware that concepts like *equiangular* and *equilateral* needed to be defined.

Finally, she was aware that the definitions she had come up with may lack adequate clarity, and may even have inconsistencies that she had not been able to detect. But the definitions given above in (1)-(10) and the axioms in A-E served as a good starting point for constructing an axiomatic theory of geometric objects.

Recall that a ***proof*** is a set of statements composed of (a) premises, (b) conclusions, and (c) steps of reasoning (called derivation) from the premises to the conclusion (the PDC structure presented in earlier chapters). The derivation shows the conclusions to be the logical consequences of the premises. In Axiomatic Proofs, the premises are axioms and definitions.

To get a sense of how Axiomatic Proofs work, let us take as an example Daza's proof for the claim that "no two straight lines can meet or cross at two distinct points."

***An Axiomatic Proof*:**

To prove: No two straight lines can meet or cross at two distinct points.

Premises:

P1: A straight line is the shortest path between two distinct points. (**DEF 3**)

P2: *Unique Path*: For any two distinct points A and B, there exists one and only one shortest path between them. (**AXIOM A**)

P3: *Rejection of Logical Contradictions*: Logically contradictory propositions cannot be accepted as true in a body of knowledge. (**AXIOM E**)

Derivation: (Steps of reasoning)

S1: Assume that there exist two straight lines that meet/cross at two distinct points A and B [contrary to what we need to prove].

S2: By P1 and S1, there are two shortest paths between A and B.

S3: By S2, there are two shortest paths between A and B, and by P2, there cannot be two shortest paths between A and B.

S4: S3 contains a logical contradiction, and hence by P3, S3 cannot be accepted as true.

S5: To eliminate the logical contradiction, we must reject at least one of the premises that leads to the contradiction.

S6: We choose to reject S1.

S7: Hence, we conclude that no two straight lines can meet/cross at two distinct points.

Conclusion: No two straight lines can meet/cross at two distinct points. (**QED**)

A note on S6: To prove the statement, "No two straight lines can meet/cross at two distinct points," we chose to reject S1. Alternatively, we may choose to accept S1, reject the conclusion, and hold that there exist straight lines that meet/cross at two distinct points A and B.

To take another example of a proof, let us return to the theorem of the sum of angles in a triangle, proved as an observational generalisation in §8.4. Let us now prove it axiomatically.

An Axiomatic Proof:

To prove: The sum of angles in a triangle is equal to two right angles (= a straight angle).

Premises:

P1: When a straight line crosses two parallel lines, their internal alternate angles are congruent (i.e., they are equal in size).

Derivation:

S1: In your mind, construct a triangle ABC. It can be as in any of the triangles in *Fig. 9-1*:

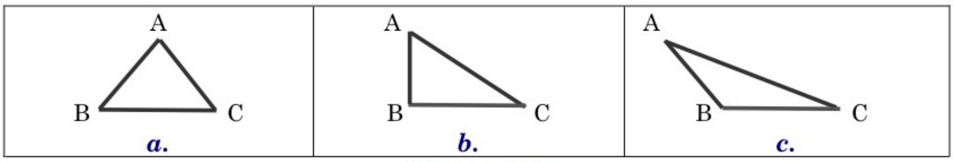


Figure 9-1

S2: Again in your mind, draw a straight line DE through A, such that DE is parallel to BC, as in *Fig. 9-2*:

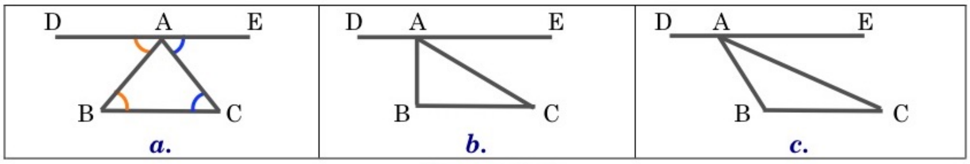


Figure 9-2

S3: By P1, ∠ DAB = ∠ ABC (marked))
∠ EAC = ∠ ACB (marked))

S4: The sum of angles of triangle ABC
= ∠ ABC + ∠ BCA + ∠ CAB
= ∠ DAB + ∠ BAC + ∠ CAE (by S3)
= a straight angle = two right angles

The same argument applies to all the figures in *Fig. 9.2*.

Conclusion: The sum of angles in a triangle is equal to a straight angle. (QED)

9.2 Diversity in the World of Geometries

Euclidean geometry — the kind of geometry we learn in school — is a branch of mathematics, unlike the Sumerian and Egyptian geometries which were branches of science. Given that theories in mathematics are axiomatic systems, it follows that if we change any of the postulates in Euclidean geometry, we get a different theory of geometry. That means that there are different kinds of geometry, depending on the axioms and definitions that make up the system.

To get a sense of what that means, let us look at diagrams. Which academic discipline studies the properties of objects such as those given in *Fig. 9-3* ?

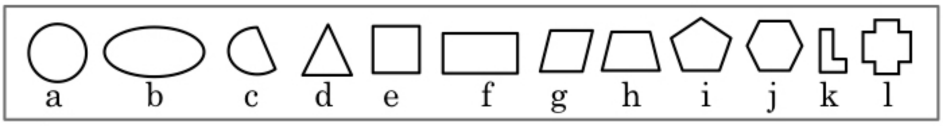


Figure 9-3

The answer is obvious: GEOMETRY.

Now take the shapes in *Fig. 9-4*:

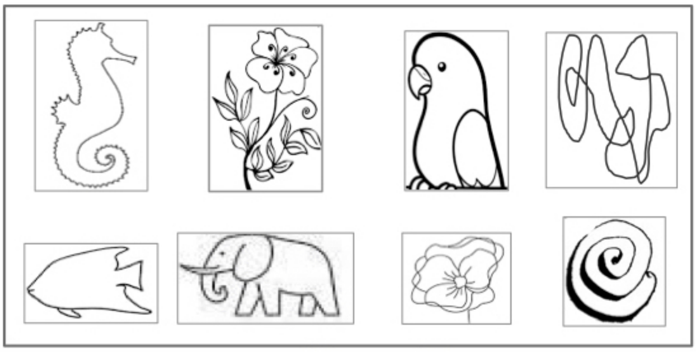


Figure 9-4

In the journey we have undertaken so far through the terrain of geometry, we have been constructing a theory of geometric shapes, using concepts of SPACE, SUBSPACE, POINT, LINE, FIGURE, and SO ON. The next step in that journey would be to ask: Do the shapes in *Fig. 9-4* come under geometry?

This question might strike you as too difficult to engage with. And if you do feel that way, don't worry: that would be challenging even for many who have Master's and PhD's in mathematics. However, it is important to face such challenges and wrestle with them in order to learn.

A bit of the historical background first. The first part of the words *geography and geometry* have the same root: *geo-* means earth. *Geo-* appears in *geology, geoid*, and *geodesic* as well. *Geometry* originally referred to '*measuring* the earth' (*-metry* refers to measurement, as in *psychometry* and *photometry*, derived from *meter*), while *geography* originally 'referred to '*describing* the earth' (*-graphy* refers to representing, as in *photography, videography,* and *spectrography*, derived from *graph*, as in *sonograph* and *spectrograph*). As a discipline, geometry emerged as the mathematics for measurement of the earth,and for the practical needs of pursuits such as architecture and carpentry.

Also note that figures (as in geometric figures) and maps (as in street maps and geographical maps) are visual ***representations***. A map in geography is a ***representation of a region on earth***, (e.g., maps of the states of a country (*Fig. 9-5A*) or street maps (*Fig. 9-5B*)):

__Figure 9-5A__

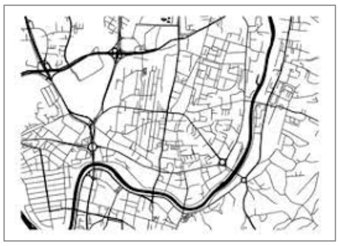


__Figure 9-5B__

And a figure in geometry is a ***representation of a mathematical object***. While regions on the earth are part of the universe we live in, geometric objects in mathematics are abtract objects created by the human mind.

Geometric objects exist in abstract ***space***. Newton's theory of gravity and motion uses the space set up in Euclidean geometry. This space is ***flat*** and ***rigid***. Einstein's relativity theory, on the other hand, uses the space set up in Riemannian geometry. This space can be ***curved***. And in Einstein's theory, mass causes a curvature in that space.

In the geometries of both Euclid and Riemann, the space is ***gradient***. What this means is, any line or region in this space can be divided and sub-divided such that the process of division never ends. But there are other geometries, called ***discrete geometries*** such as the ones used in computer animation, in which the space is ***discrete***. In a discrete geometry, the process of dividing a line or a region cannot be without end. At some stage, we get to the smallest indivisible entity, where the division has to stop. This is similar to the idea of fundamental particles in the modern theory of matter: if we keep dividing matter, we ultimately get to these indivisible particles, and the division has to stop.

9.3 Points, Lines, and Regions

Let us explore a new kind of geometry with the concepts of points, lines, and regions. These are abstract objects that exist in our imagination, but we may visualise them in terms of pictures like those in *Fig. 9-6*:

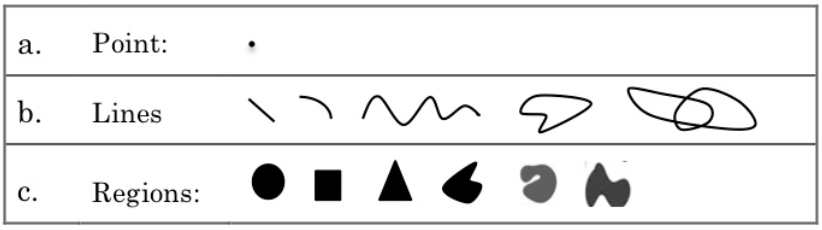


__Figure 9-6__

We may think of a ***point*** as an imaginary object constructed by pressing on an imaginary surface (e.g., white board, paper) with a marker (e.g., a pencil, a pen, a white board maker) and lifting it, without dragging it. In this sense, a *point* is a *dot* on a *surface*. A ***line*** would be an object constructed on a surface with a marker, dragging it, and then lifting it. And a ***region*** would then be what is inside a line that returns to its starting point without crossing itself. Let us state these as postulates (POSs), as follows:

POS-1: A ***point*** is an ***atomic unit*** (i.e., the smallest indivisible unit) in geometry, with no length, breadth, or thickness. A ***line*** has length but no breadth, or thickness. A ***region*** has length and breadth, and hence area, but no thickness.

POS-2: ***Lines*** and ***regions*** are composed of points.

POS-3: The ***magnitude*** (size) of a geometric object — the length of a line, the area of a region, or the length of its edge or diameter or diagonal — is the number of points it is composed of. This means that the size of X is determined by counting the number of points X has, and is expressed as a natural number (counting number).

This set of postulates is a reasonably good nucleus to start building a theory. We will refer to a geometry that includes POS-3 as Natural Number Geometry (NNG), meaning that it allows magnitude to be expressed as a natural number that lends itself to counting, as distince from a Rational Number Geometry (RNG) that lends itself to measuring but not to counting.

While POS-1 is common to NNG and RNG, POS-2 and POS-3 are specific to NNG, distinguishing it from RNG. The consequences of these two postulates, especially POS-3, are already far-reaching. To get an intuitive sense of those far-reaching logical consequences (theorems), consider this question:

Can every line be divided into two equal parts?

If you remember your high school geometry, you can see that the answer in RNG (e.g., Euclidean geometry) is 'yes'. But the answer in NNG is 'no'!

Exercise

Prove that in a theory of geometry whose postulates include (1)-(3), there exist lines that cannot be divided into two equal parts.

Let us go a bit further. Suppose we add the following postulates to NNG:

POS 4: A line is composed of at ***least two adjacent*** points.

POS 5: A ***straight line*** with two end points is the shortest path between those points.

POS 6: A region is composed of at ***least three adjacent points*** which are not co-linear (i.e., not on the same straight line).

POS 7:. Two points are ***adjacent*** if there is no other point between them. (When two points are adjacent, we say that they are ***neighbours***.)

To have a rudimentary understanding of these postulates, it would be a good idea to meditate on them. To have an intuitive sense of the postulates, let us represent points as circles. This is just a pictorial convention: we are not assuming that points can have shape — circular, elliptical, triangular, rectangular, wavy, and so on.

A line can now be represented as in *Fig. 9-7*:

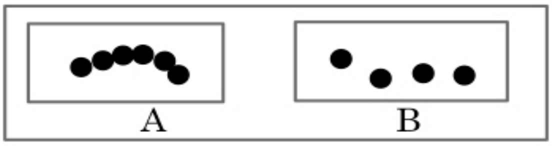


Figure 9-7

In these representations, the adjacency of two points is expressed by their touching each other. In *Fig. 9-6A,* the points are adjacent. In contrast, no two points in *Fig. 9-6B* are adjacent.

Instead, suppose we represent the relation, "X and Y are adjacent," with a connecting line, as in *Fig. 9-7A* and *Fig. 9-7B*:

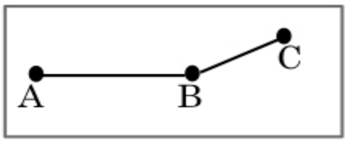


Figure 9-8A

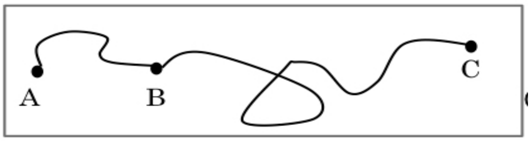


Figure 9-8B

In this notation , whether the line connecting two points is straight or curved does not matter. The distance between the dots on paper is also irrelevant. *Fig. 9-7A* and *Fig. 9-7B* express exactly the same information.

In what is called ***graph theory*** or ***network theory*** in mathematics, the points in such diagrams are called ***nodes*** or ***vertices*** (singular ***vertex)***, and the lines connecting them are called ***arcs*** or ***edges***. Before you proceed, take a look at our thirteen-page introduction to graph theory, "Mathematising theories in language sciences and life sciences," downloadable from https://www.thinq.education/post/mathematising-theories-in-language-sciences-and-life-sciences . You might also read the seventeen-page write up, "An Introduction to Graph Theory," at https://builtin.com/machine-learning/graph-theory, or watch a 100-minute YouTube video, "A gentle introduction to network science: Dr Renaud Lambiotte, University of Oxford," at https://www.youtube.com/watch?v=L6CqqlILBCI

9.4 Graph Theory, Maps, and Colour Theorems

Suppose we are asked to colour maps in such a way that no two neighbouring regions have the same colour. How many colours will we need?

What is called the 'four-colour theorem' in mathematics says that we don't need more than four colours to colour any two-dimensional map in such a way that ***neighbours*** (regions sharing a boundary line, not just a single point) don't have the same colour.

The problem is sometimes called 'Guthrie's problem', because it was Francis Guthrie who first articulated this conjecture in 1852.

9.4.1 Four-Colour Conjecture

Suppose a teacher/facilitator draws a random figure on the blackboard, like the one in Fig. 9-9A, and the following dialogue ensues. (F: facilitator)

F: Imagine that *Fig. 9-9A* is a map, and you have to colour it.

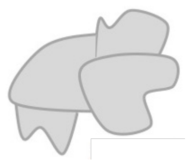


Figure 9-9A

The colouring has to obey TWO CONDITIONS:

Identical Colour Prohibition (ICP)

No two neighbours can have the same colour.

Minimal Colour Condition (MCC)

We must use the minimum number of colours.

After colouring, *Fig. 9-9A* would look as in *Fig. 9-9B:*

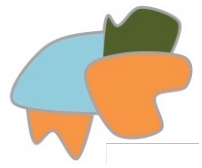


Figure 9-9B

Now take the map in *Fig. 9-10A*. In it, the adjacent pairs of regions are 1 and 2; 1 and 3; 2 and 4; and 3 and 4.

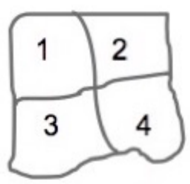


Figure 9-10A

These pairs of regions share a boundary line. Neither 1 and 4, nor 2 and 3 are neighbours, as they do not share a border, even though they share a point of intersection.

How many colours do we need to colour the map in *Fig. 9-10A*?

Clearly, the colours of 1 and 2 cannot be the same. Those of 1 and 3 cannot be the same either. But 2 and 3 can be the same. Similarly, 1 and 4 can be the same. So a possible way to colour the map is as in *Fig. 9-10B*. So **two colours are sufficient for this map.**

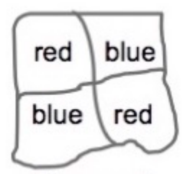


Figure 9-10B

What then is *the minimal number of colours* needed for colouring the *population* of maps — that is, *any map one can ever think of*?

The map in *Fig. 9-10B* proves that there are maps that require at least two colours. Are there maps that require at least three colours? Try drawing maps with the minimum number of lines, but requiring at least three colours.

It is important here to pay attention to the notion 'adjacent' as we have defined it. It is not difficult to come up with maps that require three colours. One such map is given in *Fig. 9-10C*. But it would be good for you to come up with such maps on your own.

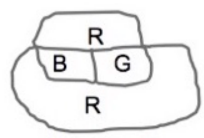


Figure 9-10C

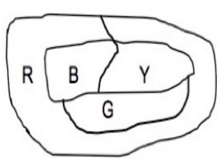


Figure 9-10D

Are there maps that require **at least four colours**?

This task is hard, but it is possible to come up with maps that require four colours, like the one in *Fig. 9-10D*. Draw more such maps on your own, to train your mind.

Are there maps that require **more than four colours?**

It is important to make a serious attempt to look for maps that require more than four colours, both mentally and on paper, both individually and collectively.

At the end of the search, chances are that your conclusion would be:

> "We have explored a large random sample of maps that obey ICP and MCC. None of them require more than four colours."

This suggests the following conjecture:

Conjecture 1: *No map that obeys ICP and MCC requires more than four colours.*

In scientific inquiry, we can give a proof of this conjecture along the following lines, and state the conclusion (the *therefore* sentence):

Conclusion in scientific inquiry:

We have explored a large random sample of maps that obey ICP and MCC.
None of those maps requires more than four colours.
Therefore, until we find a map that requires five or more colours,
it is reasonable to conclude that Conjecture 1 is true.

In mathematical inquiry, this proof is not valid, as we have not ruled out the possible existence of a five-colour map with complete certainty. Hence:

Conclusion in mathematical inquiry:

We have explored a large random sample of maps that obey ICP and MCC.
None of those maps requires more than four colours.
Therefore, we conclude that Conjecture 1 is highly plausible.

As in scientific inquiry, looking for counterexamples is an important preoccupation in mathematical inquiry. However, while the absence of counterexamples in a large random representative is sufficient to prove an observational generalisation in science, it can only establish a high degree of plausibility in mathematics. This means that it is only a ***plausible conjecture***, ***not yet a theorem***. After Guthrie discovered the four-colour conjecture, it took the mathematics community a hundred years to come up with a proof, and establish it as a theorem.
(http://www.ams.org/notices/200811/tx081101382p.pdf)

The exercises below would be useful in developing the mental discipline needed for restricting yourself to the absolutely minimum number of lines when creating the maps of specified kinds.

Exercise 1

A. Draw a circle C1.

- i) Draw a straight line in C1 such that it becomes a map that requires exactly two colours.
- ii) Draw two straight lines in C1 such that it becomes a map that requires exactly three colours.
- iii) Draw two straight lines in C1 such that it becomes a map that requires only two colours.
- iv) How many lines (minimum) do you need to draw to require four colours?

B. Draw circle C1, and another circle C2 inside C1.

- i) Draw a straight line in C2 such that C1+C2 becomes a map that requires three colours.
- ii) Extend the line in C2 till it touches the circumference of C1 on both sides. How many colours does this map need?
- iii) Draw a minimum number of straight lines such that C1+C2 becomes a map that requires four colours.

9.4.2 Maps with Straight Lines

Having established that there exist maps obeying ICP and MCC that require two, three, and four colours, the next question is:

> Are there general properties that characterize the categories of maps that call for exactly two colours, exactly three colours, and exactly four colours.

F: Take a sheet of paper, and follow this procedure (as the creasologists would have done):

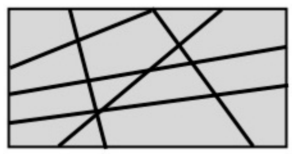


Figure 9-11A

Procedure 1:
On a sheet of paper, draw a set of straight lines such that each line extends from one edge of the paper to the opposite edge, as in *Fig. 9-11A*.

How many colours would you need to colour each region, without violating ICP and MCC?

It is easy to see that:

Conjecture 2: *Maps that are created using Procedure 1, and obey ICP and MCC, require exactly two colours.*

Once again, in order to establish the plausibility of this conjecture, it is important to look for counterexamples and make sure there are none..

F: Now let us try a different procedure:

Procedure 2:
Draw a set of straight lines where each line extends from one side of the paper or from an existing line to another side of the paper or to another line.

Here, in *Fig. 9-11B*, is an example that results from Procedure 2.

How many colours would be needed this time to colour each region?

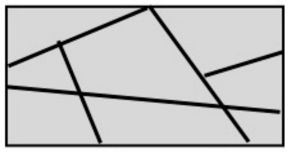


Figure 9-11B

Once again, it should be easy to see that:

Conjecture 3: *Maps that are created using Procedure 2, and obey ICP and MCC, require no more than three colours.*

Again,you need to look for counterexamples to establish the plausibility of Conjecture 3.

Exercise 2

Draw a rectangle R1.

A. Draw one line in R-1 such that the map requires exactly two colours.

B. Draw two lines in R-1 such that the map requires exactly two colours.

C. Draw two lines in R-1 such that the map requires exactly three colours.

F: Let us try yet another procedure:

Procedure 3:
Draw a set of straight lines such that the resulting map requires exactly four colours.

Notice that Procedure 3 is a simplified version of Procedures 1 and 2. Following this procedure, can you create a map that requires four colours?

Can you now create a map that requires four colours, but with the minimum number of lines?

9.4.3 Maps with Circles

F: Take a sheet of paper, and follow procedure 4 to convert it into a map:

Procedure 4: Draw a set of circles.

(Some of these circles may intersect, while others may not. If they do not intersect, some circles can be inside another, while others may not be.)

How many colours do the maps resulting from procedure 4 require?

Exercise 3

We can create a variant of procedure 4 by replacing circles with polygons.

Procedure 5: Draw a set of polygons.

How many colours will the maps resulting from this procedure require?

What did you learn from Procedure 4 and its variant in Procedure 5?

F: Let us try combining circles with straight lines.

Procedure 6: Draw a circle. From the center draw one, two, three or more lines meeting the circumference.

Procedure 7: Draw a circle and a cord. Put a set of points on the chord. From each point, draw two lines, one to each part of the circumference, as illustrated in *Fig. 9-11C*.

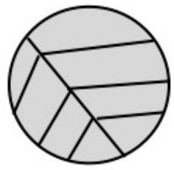


Figure 9-11C

Procedure 8: Draw a circle and a cord. Put a set of points on the chord. From each point, draw a line to one side to the circumference, the next one to the other side, and so on alternatingly, as illustrated in *Fig. 9-11D*.

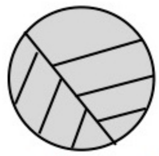


Figure 9-11D

Find out how many colours each type of maps resulting from procedures 6 – 7 requires.

9.4.4 Maps with Closed Loops

F: Let us try yet another kind of map.

Procedure 9:

Draw a closed curve, with as many self-intersections as you wish. Once you put the pen on paper, you cannot lift it until you return to the starting point.

Here is an example: *Fig. 9-12*.

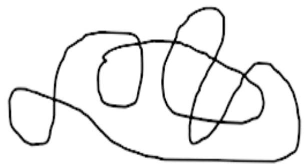


Figure 9-12

How many colours do you need for the maps created through Procedure 9?

Exercise 4

Compare the two colour maps, three colour maps, and four colour maps you have examined so far. Do you see a pattern emerging? Are there some common features that the two colour maps have which distinguish them from the other two sets of maps? Are there some common features that the three colour maps have that distinguish them from four colour maps?

9.5 What did we learn?

The geometry presented in Chapters 1-8 was the geometry of ***flat rigid continuous surfaces***, a prototypical example being Euclidean geometry, which is taught in schools. Recall that at the end of Chapter 8, the alliance of creasogists and thropologists set up the following postulates:

Postualtes in Chapter 8:

POS-1: A **LINE** is an abstract entity that has length, but no breadth or thickness.

POS-2: An **INTERSECTION** or **JOINT** between two lines is a **POINT**. If a line has finite length, its two ends are also points.

POS-3: A line between two points is straight if and only if it is the shortest path between them.

In addition, Euclidean geometry incorporates the following postulate:

POS-4: Between any two points in a line, there are infinitely many points.

[Note: The wording of the postulates is ours. Our intention is to capture the features of geometry that we think are noteworthy, not to faithfully reproduce what is given in English translations of Euclid's work.]

Of these postulates, what is particularly relevant for our purposes is POS-4. As we see it, what it says is that the so-called number line in Euclidean geometry is a line of rational numbers, which was why we called it Rational Number Geometry (RNG).

In section 9.4, we formulated an alternative set of postulates, repeated below:

POS-1: A **POINT** is an **ATOMIC UNIT** (i.e., the smallest indivisible unit) in geometry, with no length, breadth, or thickness. A **LINE** has length but no breadth, or thickness. A **REGION** has length and breadth, and hence area, but no thickness.

POS-2: **LINES** and **REGIONS** are composed of points.

POS-3: The **MAGNITUDE** (size) of a geometric object — the length of a line, the area of a region, or the length of its edge or diameter or diagonal — is the number of points it is composed of. This means that the size of X is determined by counting the number of points X has, and is expressed as a natural number (counting number).

POS 4: A line is composed of at ***least two*** **ADJACENT POINTS**.

POS 5: A **STRAIGHT LINE** with two end points is the shortest path between those points.

POS 6: A region is composed of at ***least three*** **ADJACENT POINTS** which are not co-linear (i.e., not on the same straight line).

POS 7:. Two points are **ADJACENT** if there is no other point between them. (When two points are adjacent, we say they are **NEIGHBOURS**.)

In saying that lines are composed of points, POS-2 departs from RNG. And POS-3 makes the departure even sharper. What it says is that the number line in this geometry is that of Natural Numbers (or counting numbers), which is why we called it Natural Number Geometry (NNG). In NNG, then, the length of a line can be specified by counting the number of points it is composed of, and by extension, the area of a region by counting the number of points it is composed of.

As hinted at earlier in this Chapter, the consequences of POS-3 are far-reaching. To illustrate, consider the Pythagoras theorem which says that the square of the hypotenuse of a right-angle triangle is equal to the sum of the squares of the other two sides ($A^2 + B^2 = C^2$). Now consider the lengh of the hypotenuse (C) of a right-angle triangle whose other two sides (A and B) are of one meter length each. Since the square of one is one, the sum of the squares is two. Hence by this theorem, the length of the hypotenuse is the square root of two (C = square root of ($A^2 + B^2$)). Now, the square root of 2 is an irrational number. Therefore it is not a countable number. A logical consequence of that result is that in NNG, the length of the hypotenuse of such a right-angle triangle cannot be calculated

Another difference between NNG and RNG appears in the relation between the diameter of a circle and its circumference, and given that relation, whether or not one can be calculated from the other. The formula for the relation tells us that the length of the circumference is π times the diameter. However, π, like the square root of 2, is an irrational number, and hence, not countable. This means that in NNG, it is not possible to calculate the length of the diameter of a circle from the length of its circumference, nor can the length of the circumference from the length of the diameter.

Take another example. Consider a line segment made of an odd number of points. As pointed out earlier in this chapter, such a line segment cannot be divided into two equal parts in NNG, unlike in RNG, where every line segment can be divided into two equal parts.

One of the reasons for taking this journey into a comparison of NNG and RNG was to make clear that Euclidean geometry — the geometry we study in school — is not the only geometry in the world of mathematics. There are other kinds of geometries, like spherical geometry, projective geometry, taxicab geometry, and so on. Such multiplicities of geometries is the result of different conceptualisations of points, lines, regions, and space.

To get a sense of the range of alternative geometries, consider the following postuates:

POS-I: No straight line, even when extended indefinitely in either direction, will meet itself.

POS-II: Every straight line, when extended, will meet itself.

POS-I yields a flat surface geometry. Newton's theory of gravity and motion is built on this geometry. In this geometry, the sum of angles of a triangle is two right angles. And no two straight lines can intersect at more than one distinct point.

On the other hand, POS-II yields a spherical surface geometry. Einstein's theory of gravity and motion is built on this geometry. In this geometry, the sum of angles in a triangle is more than two right angles, up to three right angles. And any two straight lines intersect at two distinct points.

In order to get a sense of the diversity of geometries in mathematics, you may wish to do an Internet search for the different kinds of geometries (spherical geometry, projective geometry, taxicab geometry, pixel geometry, fractal geometry, …). Better still, try to get hold of a copy of Ian Stewart's book, *Flatterland* (2001). If you do not have the time to read the book, watch the YouTube video, "Non-Euclidean Geometry & the Shape of Space – Tony Weathers – May 2, 2013," at https://www.youtube.com/watch?v=jJq1vZbOrwI. It will give you a clearer sense of the concept of Euclid's postulates and the alternative postulates.

The purpose of introducing you to some of these ideas was to strengthen your potential for ***conceptualisation***, academic ***imagination*** and ***intuition***. These are the three primary learning outcomes that this chapter aims at. We may now add the following learning outcomes to the different strands of capacity building that the chapter aims at:

- ~ discovering patterns and formulate them as conjectures;
- ~ looking for counterexamples to demonstrate the plausibility of conjectures;
- ~ coming up with and formulate axioms and definitions;
- ~ coming up with and formulate proofs;

and an understanding of

- ~ the distinction between mathematical and scientific proofs.

CHAPTER 10:
LOOKING BACK: WHAT DID WE LEARN?

You can look back only if you have gone through the journey of reading Chapters 1-9 in this book. ☺

Much of the information in these chapters must have already been familiar to you from what you have learnt in school, like some of the concepts and theorems of (Euclidean) geometry. You must have learnt the concepts of triangle, right-angled triangle, equilateral triangle, isosceles triangle, square, rectangle, parallelogram, quadrilateral, pentagon, hexagon, polygon, equilateral polygon, equiangular polygon, regular polygon, diagonal, line, point, parallel lines, intersection, angle, vertex, ellipse, circle, diameter, radius, arc, and circumference. You must also have been exposed to statements like the angle sum theorem, the Pythagoras theorem, some of the circle theorems, the formulae for the circumference and area of circles, the formulae for the area of triangles, rectangles, parallelograms, quadrilaterals, and so on.

So what did you learn here that was new? What makes this book different from the books on geometry that you are familiar with?

It is likely that you view the geometry (and other branches of mathematics) that you have learnt as a body of ***facts*** that can be applied to various problems. Take a moment to think about that. If mathematics is a body of ***facts***, and the angle sum theorem is a fact, and you ***know*** that fact, why should you learn to ***prove*** the angle sum theorem? You know for a fact that stubbing your toe is painful, so why should you prove that stubbing your toe is painful? Have you thought about why you need to prove what you already know?

Here is why. In mathematics as a branch of academic knowledge, theorems are not facts about the world. They are rationally justified conclusions, proved on the basis of assumptions that we call axioms and definitions. So, we might *feel* that a certain conjecture is true. But for it to be admitted as knowledge in mathematics, it must be proved.

Now, when you think of math, what first comes to mind is calculation. But we have not explored any calculation in this book. There are two reason for this:

1. For many learners, calculations are really hard. And when they are compelled to calculate, they become afraid of mathematics, and even begin to hate it: they develop mathophobia. We would like to help learners to develop a love of math, and to discover how joyful doing math can be.

2: We believe that it is important for learners to develop the ability to think conceptually, and to reason well. And this is best done using statements expressed in words before learners are introduced to calculation, which use symbols, formulae, and equations.

Much of what we have talked about in this book has to do with how mathematicians think, and how they reason. So this book is about ***mathematical thinking and reasoning***.

We leave you with the hope that you will make this way of thinking and reasoning a habit of your mind, extend it to other academic (and non-academic) domains, and enjoy doing so.